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O. Neugebauer J. D. Tamarkin O. Veblen

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Mathematical Reviews

Vol. 5, No. 10 NOVEMBER, 1944 Pages 253-280

HISTORY

Miller, G. A. An eighth lesson in the history of mathematics. Nat. Math. Mag. 18, 261–270 (1944). [MF 10577]

Richards, John F. C. A new manuscript of a rithmomachia. Scripta Math. 9, 169–183, 256–264 (1943). [MF 10262]

This is a continuation of an article in the same Scripta 9, 87–99 (1943); these Rev. 5, 57.

Boyer, Carl B. Zero: the symbol, the concept, the number. Nat. Math. Mag. 18, 323–330 (1944). [MF 10691]

Boyer, Carl B. Fundamental steps in the development of numeration. Isis 35, 153–168 (1944). [MF 10800]

The article consists chiefly in a summary of the standard literature on number writing. The main purpose, however, is "to indicate a reevaluation of the over-vaunted principle of position or of local value and, incidentally, a justification of the much-maligned Greek alphabetic or Ionian system of numeration." The argument of the author is based essentially on a comparison of the degree of clumsiness in expressing integers. He writes, for example, the modern form of multiplying 4506 by 23 using Babylonian, Egyptian, Greek, etc., symbols, and finds that Egyptian or Babylonian characters are much more awkward than single letters. Historically none of these operations would have been carried out in the form given by the author. The fact that the place value notation avoids all difficulties with fractions and therefore has been adopted by the astronomers seems to be of only secondary importance to the author.

O. Neugebauer (Providence, R. I.).

Reidemeister, Kurt. Mathematik und Logik bei Plato. Hamburger Math. Einzelschr. 35, 20 pp. (1942). [MF 9807]

This paper constitutes an attempt to analyze Plato's philosophy and dialectic by means of his concept of existence as revealed in his attitude towards mathematical problems.

O. Neugebauer (Providence, R. I.).

Dehn, Max. Mathematics, 300 B.C.–200 B.C. Amer. Math. Monthly 51, 25–31 (1944). [MF 10065]

Dehn, Max. Mathematics, 200 B. C.–600 A.D. Amer. Math. Monthly 51, 149–157 (1944). [MF 10165]

Rufus, W. Carl. Greek astronomy—its birth, death, and immortality. J. Roy. Astr. Soc. 38, 143–153 (1944). [MF 10331]

Rosenblatt, Alfred. Copernicus' position in the history of science. Revista Ci., Lima 45, 409–442 (1943) = Actas Acad. Ci. Lima 6, 165–198 (1943). (Spanish) [MF 10099]

Karpinski, Louis C. The progress of the Copernican theory. Scripta Math. 9, 139–154 (1943). [MF 10259]

Sleight, E. R. John Napier and his logarithms. Nat. Math. Mag. 18, 145–152 (1944). [MF 9955]

Green, H. Gwynedd and Winter, H. J. J. John Landen, F.R.S. (1719–1790)—mathematician. Isis 35, 6–10 (1944). [MF 10178]

Bell, E. T. Gauss and the early development of algebraic numbers. Nat. Math. Mag. 18, 188–204 (1944). [MF 10168]

Bell, E. T. Gauss and the early development of algebraic numbers. Nat. Math. Mag. 18, 219–223 (1944). [MF 10218]

Kagan, V. F. The great scientist N. I. Lobačevskii and his place in peaceful science. Vestnik Akad. Nauk. SSSR 1943, no. 7–8, 44–83 (1943). (Russian) [MF 10029]

Alexandroff, P. S. Lobačevskii and the Russian civilization. Vestnik Akad. Nauk SSSR 1943, no. 11–12, 52–62 (1943). (Russian) [MF 10689]

Rossell Soler, Pedro A. Obituary: Claro Cornelio Dassen. An. Soc. Ci. Argentina 135, 37–52 (1943). (Portuguese) [MF 9983]

A bibliography of Dassen's papers is given on pp. 59–66.

Kasner, Edward. Obituary: Thomas Scott Fiske. Science (N.S.) 99, 484–485 (1944). [MF 10680]

Koopman, Bernard Osgood. Obituary: William Fogg Osgood—In memoriam. Bull. Amer. Math. Soc. 50, 139–142 (1944). [MF 10122]

Cartan, Élie. Notice nécrologique sur Georges Giraud. C. R. Acad. Sci. Paris 216, 516–518 (1943). [MF 10051]

Montel, Paul. La vie et l'oeuvre d'Émile Picard. Bull. Sci. Math. (2) 66, 3–17 (1942). [MF 10461]

Crathorne, A. R. Obituary: Henry Lewis Rietz. In memoriam. Ann. Math. Statistics 15, 102–108 (1944). [MF 10245]

Smith, C. D. Obituary: Henry Lewis Rietz, 1875–1943. Nat. Math. Mag. 18, 182–184 (1944). [MF 10167]

Bailey, W. N. Francis John Welsh Whipple. J. London Math. Soc. 18, 249–256 (1943). [MF 10880]

NUMBER THEORY

Salem, R. A remarkable class of algebraic integers. Proof of a conjecture of Vijayaraghavan. Duke Math. J. 11, 103–108 (1944). [MF 10151]

We shall say that the real number θ belongs to the class C if $\theta>1$ and if there is a real number $\lambda=\lambda(\theta)\neq 0$ such that

$$\sum_{n=0}^{\infty} \sin^2 \pi\lambda\theta^n < \infty.$$

C. Pisot [Ann. Scuola Norm. Super. Pisa (2) 7, 205–248 (1938)] has shown that C is identical with the class of algebraic integers θ ($\theta>1$) whose conjugates have moduli less than 1. The author discusses the distribution of C in the interval $(1, \infty)$ and proves that it is closed (the limit point at infinity being excluded). Since C is denumerable, it follows that it is nowhere dense and not dense-in-itself, results conjectured by T. Vijayaraghavan [Proc. Cambridge Philos. Soc. 37, 349–357 (1941); these Rev. 3, 274]. It is also shown that C (which is reducible) has nonempty derived sets of any finite order.

There are interesting applications to trigonometrical series. For example, let $P(\xi)$ be the perfect set of Cantor type and constant ratio of dissection ξ. The numbers ξ $(0<\xi<\frac{1}{2})$ for which $P(\xi)$ is a set of multiplicity form an open set. *D. C. Spencer* (New York, N. Y.).

Chabauty, Claude. Sur les solutions de certaines équations diophantiennes en nombres algébriques, en particulier en entiers algébriques, de degré borné. C. R. Acad. Sci. Paris 217, 127–129 (1943). [MF 10638]

Let V be an algebraic curve whose field of coefficients is an algebraic number field K_0. A point on V is termed algebraic of degree h if its coordinates generate over K_0 an extension of degree h. Let g be the genus of V and ρ the arithmetic rank of the period matrix. The author announces, among other results, that there exists only a finite number of points of degree h provided $h\leqq g-\rho$ for a pure period matrix. *O. F. G. Schilling* (Chicago, Ill.).

Pierre, Charles. Sur le théorème de Fermat $a^n+b^n=c^n$. C. R. Acad. Sci. Paris 217, 37–39 (1943). [MF 10631]

Abel and Legendre proved that, if Fermat's theorem is true for a prime $n>2$ (with a, b, c relatively prime), then we must have $a+b=C^n/\eta$, $c-a=B^n/\eta'$, $c-b=A^n/\eta''$, where A, B, C are divisors of a, b, c, respectively, and η, η', η'' are, respectively, n or 1 according as c, b, a are divisible by n or not. The writer proves that $\eta c/C$, $\eta' b/B$ and $\eta'' a/A$ are quadratic residues of one another. As a consequence it is shown that one of a, b, c must be divisible by 4. *I. Niven* (Lafayette, Ind.).

Roussel, André. Remarques sur un énoncé de Fermat. C. R. Acad. Sci. Paris 217, 39–41 (1943). [MF 10632]

Six statements in analysis equivalent to Fermat's last theorem are given. Other statements of this sort were given by S. Wachs [C. R. Acad. Sci. Paris 211, 55–57 (1940); these Rev. 3, 67]. *I. Niven* (Lafayette, Ind.).

O'Connor, R. E. and Pall, G. The construction of integral quadratic forms of determinant 1. Duke Math. J. 11, 319–331 (1944). [MF 10670]

The authors prove that there exists a quadratic form in 24 variables with integer coefficients and determinant unity which does not represent 1 and 2. They also construct a form in 40 variables not representing 1, 2 and 3. The question whether there exist for every k forms with determinant unity and integer coefficients whose minimum is k is left open for $k>3$. *P. Erdös* (Lafayette, Ind.).

Davenport, H. The minimum of a binary cubic form. J. London Math. Soc. 18, 168–176 (1943). [MF 10369]

The object of the paper is to give simple proofs of two theorems of Mordell. These theorems may be stated as follows, where $f(x, y)$ is a binary cubic form with real coefficients and discriminant D. Theorem: there exist integers x, y not both zero for which $|f(x, y)|\leqq(-D/23)^{\frac{1}{4}}$ or $(D/49)^{\frac{1}{4}}$ according as $D<0$ or $D>0$. The strict inequality holds except when f is equivalent to $(-D/23)^{\frac{1}{4}}(x^3-xy^2-y^3)$ or $(D/49)^{\frac{1}{4}}(x^3+x^2y-2xy^2-y^3)$. *B. W. Jones.*

Mordell, L. J. Rational points on cubic curves and surfaces. Amer. Math. Monthly 51, 332–339 (1944). [MF 10679]

This is an exposition of various methods of solving in integers certain homogeneous cubic equations in three and four variables. This includes a description of some of the work of the author and B. Segre [J. London Math. Soc. 18, 24–31, 31–34, 43–46 (1943); these Rev. 5, 141, 154]. *B. W. Jones* (Ithaca, N. Y.).

Gage, Walter H. An arithmetical identity. Trans. Roy. Soc. Canada. Sect. III. 37, 9–11 (1943). [MF 9973]

A new arithmetical identity of a type similar to those obtained by Liouville is derived in this paper by the well-known method of symmetry. The identity is then applied to obtain various recursion formulas for the class-number functions $F(n)$, $G(n)$, $E(n)$. One such formula is the following: $F(m)-F(m-1^2)+22F(m-4^2)-F(m-3^2)+2F(m-8^2)-F(m-5^2)+\cdots=0$, where $F(n)$ is the number of uneven classes of quadratic forms of determinant $-n$ and $m\equiv 3$ (mod 8). *H. W. Brinkmann* (Swarthmore, Pa.).

Walfisz, Arnold. On the class-number of binary quadratic forms. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 11, 57–71 (1942). (English. Russian summary) [MF 10282]

The chief object of this paper is the proof of the following two results obtained from various class number relationships:

$$\limsup_{k\to\infty} \frac{L_k}{\log\log k} \geqq e^C$$

and

$$L_k^{-1}=\Omega(\log\log k)^{\frac{1}{2}},$$

where $L_k=\sum_{n=1}^{\infty}(-k|n)n^{-1}$, $(-k|n)$ being the Kronecker symbol of quadratic residuacity, C is Euler's constant and k is positive, square-free except perhaps for a factor 4 and identical to 3 (mod 4) or 4, 8 (mod 16). The symbol Ω apparently means that there is a constant A and an increasing sequence of positive integers k such that $L_k^{-1}\geqq A(\log\log k)^{\frac{1}{2}}$. The former of the two results was obtained under a certain assumption by Littlewood [Proc. London Math. Soc. (2) 27, 358–372 (1927)]. A direct application to the class number function is provided by the relationship $h(k)=\sqrt{k}\,L_k/\pi$ if $k>4$, where $h(n)$ is, for $n>4$, the number of primitive classes of forms $ax^2+bxy+cy^2$ with $b^2-4ac=-n$. *B. W. Jones* (Ithaca, N. Y.).

Walfisz, Arnold. On the additive theory of numbers. X. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 11, 173–186 (1942). (English. Russian summary) [MF 10287]

[For the last paper of the series, cf. the same Trav. 9, 75–96 (1941); these Rev. 4, 132.] von Sterneck's table giving the decomposition of integers $n\leqq 40{,}000$ into cubes was used by Dickson to show the universality of certain cubic forms. In order to show that the form $20x_1^3+x_2^3+\cdots+x_9^3$ is universal it is necessary to know that all integers n such that $40{,}000<n\leqq 61{,}496$ are sums of six cubes. This is beyond von Sterneck's table. It is the main object of this paper to show that the form $20x_1^3+x_2^3+\cdots+x_9^3$ is universal using Dase's table published by Jacobi [J. Reine Angew. Math. 42, 41–69 (1851)] giving the decomposition of integers $n\leqq 12{,}000$ into cubes. It is also shown that to prove that all integers are sums of nine cubes it is only necessary to have a table for $n\leqq 3{,}000$. Finally, some exceptions are noted to Dickson's statement [Bull. Amer. Math. Soc. 39, 701–727 (1933)] that each of the forms $x_1^3+x_2^3+x_3^3+2x_4^3+2x_5^3$, $x_1^3+x_2^3+2x_3^3+2x_4^3+2x_5^3$ represents all integers $n\leqq 40{,}000$. *R. D. James.*

Hurwitz, S. On a class of functions suggested by the zeta of Riemann. Ann. of Math. (2) 45, 340–346 (1944). [MF 10270]

Let p denote a nonnegative integer. Put $\log_0 x=x$, $\log_{p+1} x=\log(\log_p x)$, and define

$$\zeta_p(s)=\sum_{n=V+1}^{\infty}\left\{\prod_{0}^{p-1}\log_i n\right\}^{-1}(\log_p n)^{-s},$$

where $V=[e_p(0)]$, $e_{p+1}(x)=\exp(e_p(x))$, $e_0(x)=x$. Thus $\zeta_0(s)=\zeta(s)$, the Riemann zeta-function. The writer proves that $\zeta_p(s)$ is regular everywhere except at $s=1$, where it has a simple pole with residue 1. Various integral formulas are obtained and it is proved that $\zeta_p(s)$ has an infinite number of negative zeros (exact values not determined). Finally some theorems on the asymptotic behavior of $\zeta_p(s)$ are proved. *L. Carlitz* (Durham, N. C.).

Wintner, Aurel. Random factorizations and Riemann's hypothesis. Duke Math. J. 11, 267–275 (1944). [MF 10665]

If x is a random sequence of signs $(\pm,\pm,\pm,\cdots)$ and if in the product

$$1/\zeta_x(s)=\prod_p(1\pm p^{-s})$$

the prime numbers are arranged in ascending order, then, for almost all x, this product and the corresponding series

$$1/\zeta_x(s)=\sum_{n=1}^{\infty}\mu_x(n)n^{-s}$$

are convergent for $\sigma>\frac{1}{2}$ but are not continuable anywhere beyond the line $\sigma=\frac{1}{2}$. *S. Bochner* (Princeton, N. J.).

Wintner, Aurel. Eulerian products and analytic continuation. Duke Math. J. 11, 277–285 (1944). [MF 10666]

The author exhibits a multiplicative function $g(n)$ for which the Euler product

$$\prod_p\left(1+\sum_{k=1}^{\infty}g(p^k)/p^{ks}\right)$$

converges in a larger half-plane than the corresponding series

$$\sum_{n=1}^{\infty}g(n)/n^s,$$

thus controverting the classical cases in which the series invariably converges at least as far as the Euler product. *S. Bochner* (Princeton, N. J.).

Wintner, Aurel. The singularities in a family of zeta-functions. Duke Math. J. 11, 287–291 (1944). [MF 10667]

The author discusses the function

$$\zeta_\alpha(s)=\prod_p(1-\alpha p^{-s})^{-1}$$

for real parameters α. In particular, he shows that $(\zeta-1)^\alpha\zeta_\alpha(s)$ is regular and nonvanishing on the line $\sigma=1$ and that the Riemann hypothesis for the classical series $\zeta_1(s)$ implies the same hypothesis for the other series. *S. Bochner* (Princeton, N. J.).

Hua, Loo-keng and Min, Sze-hoa. An analogue of Tarry's problem. Acad. Sinica Science Record 1, 26–29 (1942). [MF 8828]

Asymptotic formulae are found for the number of solutions of the system of congruences (analogous to the equations of Tarry's problem)

$$x_1^h+\cdots+x_s^h\equiv y_1^h+\cdots+y_s^h,\quad 1\leqq h\leqq k \pmod{p^n},$$

where p is a prime, $p>k$, $s\geqq k\geqq 2$, $n\geqq k^2$. Since this number is $p^{-kn}S$, where

$$S=\sum_{a_k=1}^{p^n}\cdots\sum_{a_1=1}^{p^n}\left|\sum_{x=1}^{p^n}e^{2\pi i(a_kx^k+\cdots+a_1x)/p^n}\right|^{2s},$$

the problem reduces to estimating sums

$$\sum\exp(2\pi if(x)/p^n)$$

for various polynomials $f(x)$. Precise results are stated, but proofs will appear later. *G. Pall* (Montreal, Que.).

Hua, Loo-keng and Min, Sze-hoa. On a double exponential sum. Acad. Sinica Science Record 1, 23–25 (1942). [MF 8827]

A proof is sketched of the following theorem. Let K be a finite field with p^m elements; let $f(x,y)$ be a polynomial of degree $n\geqq 4$, not expressible as a polynomial in one variable over K. Then

$$\sum_{\xi,\eta}e[S(f(\xi,\eta))]/p=O(p^{m(2-2/n)}),\quad e[x]=e^{2\pi ix},$$

where ξ, η run independently over K, and $S(a)$ denotes the trace of a in K; the constant implied by O depends only on n. In particular, $\sum\sum e[f(\xi,\eta)]/p=O(p^{2-2/n})$, where ξ, η run over $1,\cdots,p$. [For the case of one variable, see Hua, C. R. Acad. Sci. Paris 210, 520–523 (1940); J. Chinese Math. Soc. 2, 301–312 (1940); these Rev. 2, 40, 347.] An improved result is stated for $n=3$. *G. Pall.*

Hua, Loo-Keng. On character sums. Acad. Sinica Science Record 1, 21–23 (1942). [MF 8826]

The following lemma is established and indications given of applications [cf. Bull. Amer. Math. Soc. 48, 726–730 (1942); these Rev. 4, 130]. Let k be an integer greater than 1. For each nonprincipal character $\chi(n)$ mod k,

$$\left|\sum_{a=0}^{A}\sum_{n=-a}^{a}\chi(n)\right|\leqq(A^*+1)\sqrt{k},$$

where A^* is the least positive integer satisfying $A^*\equiv A$ mod k. *G. Pall* (Montreal, Que.).

Hua, Loo-keng. On some problems of the geometrical theory of numbers. Acad. Sinica Science Record 1, 19–21 (1942). [MF 8825]

Some results are announced, the methods being stated to be refinements of those originated by Van der Corput, Titchmarsh and Vinogradow. The number of lattice points in the circle $u^2+v^2\leqq x^2$ and in the sphere $u^2+v^2+w^2\leqq x^2$ are obtained with improved error terms; some results on the order of Epstein's zeta-function are promised. The last uses the following lemma for which an immediate proof is given: a ternary cubic form is factorable into three linearly independent factors if and only if the ratio of the form and its Hessian is a constant. *G. Pall* (Montreal, Que.).

Monna, A. F. Zur Geometrie der *P*-adischen Zahlen. Nederl. Akad. Wetensch., Proc. 45, 981–986 (1942). [MF 10449]

The author shows that Blichfeldt's method in the geometry of numbers, in the form given by C. Visser [Nederl. Akad. Wetensch., Proc. 42, 487–490 (1939)], has an analogue in the n-dimensional space of all points $(x_1, \cdots, x_n)$ with P-adic coordinates $x_1, \cdots, x_n$. For this space, measure and upper densities of sets and an integral analogous to Lebesgue's are defined. *K. Mahler* (Manchester).

Monna, A. F. Zur Theorie des Maszes im Körper der *P*-adischen Zahlen. Nederl. Akad. Wetensch., Proc. 45, 978–980 (1942). [MF 10448]

A simple one-to-one mapping of the real interval R: $0\leqq x\leqq 1$ onto the set R_P of all P-adic integers (P a prime number) is defined which changes open real sets O in R into (both open and closed) P-adic sets $\bar{O}$ in R_P, such that the Lebesgue measure of O is equal to the P-adic measure of $\bar{O}$ as defined by H. Turkstra [Metrische bijdragen tot de theorie der diophantische approximaties in het lichaam der P-adische getallen, Dissertation, Amsterdam, 1938].
K. Mahler (Manchester).

Koksma, J. F. Contribution à la théorie métrique des approximations diophantiques non-linéaires. I. Nederl. Akad. Wetensch., Proc. 45, 176–183 (1942). [MF 10386]

Koksma, J. F. Contribution à la théorie métrique des approximations diophantiques non-linéaires. II. Nederl. Akad. Wetensch., Proc. 45, 263–268 (1942). [MF 10394]

A well-known theorem of Khintchine states that, if n is a positive integer, and if $\omega(x)$ is a positive function of the positive integer x such that $x[\omega(x)]^n$ tends monotonically to zero as $x\rightarrow\infty$, then the n simultaneous Diophantine inequalities

$$|\theta_\nu x-y_\nu|<\omega(x), \qquad \nu=1, 2, \cdots, n,$$

where $P(\theta_1, \theta_2, \cdots, \theta_n)$ is a given point of the space R_n, have an infinite number of integral solutions $\{x\geqq 1, y_1, y_2, \cdots, y_n\}$ for almost all points P in R_n, provided that the series $\sum_{x=1}^\infty[\omega(x)]^n$ is divergent.

The object of the papers is to generalize Khintchine's result to the case in which x can take (for each ν) an assigned sequence s_ν of integral values $f_\nu(1), f_\nu(2), \cdots$ instead of all possible integral values. The Diophantine inequalities considered become

$$|\theta_\nu f_\nu(x)-y_\nu|<\omega_\nu(x), \qquad \nu=1, 2, \cdots, n,$$

under the hypotheses: $\omega_\nu(x)$ positive and monotonic decreasing; $x\prod_1^n\omega_\nu(x)\rightarrow 0$ for $x\rightarrow\infty$; $\sum_1^\infty\prod_1^n\omega_\nu(x)$ divergent (it is also shown that these hypotheses can be transformed so as to become less stringent). Then Khintchine's result holds provided that the sequences s_ν satisfy certain conditions which would be too long to state here, but which are satisfied for very wide types of sequences. *R. Salem.*

Koksma, J. F. et Meulenbeld, B. Sur le théorème de Minkowski, concernant un système de formes linéaires réelles. I. Introduction. Applications. Nederl. Akad. Wetensch., Proc. 45, 256–262 (1942). [MF 10393]

Koksma, J. F. et Meulenbeld, B. Sur le théorème de Minkowski, concernant un système de formes linéaires réelles. II. Lemmes et démonstration du théorème 1. Nederl. Akad. Wetensch., Proc. 45, 354–359 (1942). [MF 10402]

Koksma, J. F. et Meulenbeld, B. Sur le théorème de Minkowski, concernant un système de formes linéaires réelles. III. Démonstration des lemmes 5 et 6. Nederl. Akad. Wetensch., Proc. 45, 471–478 (1942). [MF 10412]

Koksma, J. F. et Meulenbeld, B. Sur le théorème de Minkowski, concernant un système de formes linéaires réelles. IV. Démonstration du lemme 1 (fin). Remarque sur le théorème 1. Nederl. Akad. Wetensch., Proc. 45, 578–584 (1942). [MF 10424]

The authors prove the following theorem. "Let n and r be positive integers such that $1\leqq r\leqq n$. For $(n+1)/2\leqq r\leqq n$ define a number $\rho_{n,r}$ by

$$\rho_{n,r}=\frac{2^{n+1}}{(n+1)!}\left\{\frac{1}{r^r}\sum_{\mu=0}^{r}\binom{n+1}{\mu}\left(r-\frac{n+1}{2}\right)^\mu(n+1-r)^{r-\mu}\right.$$
$$\left.+\frac{r(n+1)}{n+1-r}\binom{n}{r}\sum_{\mu=r+1}^{\infty}\frac{1}{\mu r^\mu}\left(r-\frac{n+1}{2}\right)^\mu\right\},$$

and then put $\rho^*_{n,r}=\rho_{n,n+1-r}$ for $1\leqq r<(n+1)/2$, $\rho^*_{n,r}=\rho_{n,r}$ for $(n+1)/2\leqq r\leqq n$. Further denote by $L_1, \cdots, L_{n+1}$ a set of $n+1$ linear forms with real coefficients and determinant $\Delta\neq 0$. Then to every $t>2$ there exists a system of integers $x_1, \cdots, x_{n+1}$ not all zero for which these forms $L_\nu=L_\nu(x_1, \cdots, x_{n+1})$ satisfy the inequalities

$$\sum_{\nu=1}^{r}|L_\nu|\leqq 2\left(\frac{t^{n+1-r}|\Delta|}{\rho^*_{n,r}}\right)^{1/r}, \quad \sum_{\nu=r+1}^{n+1}|L_\nu|\leqq\frac{2}{t},$$

$$\left(\sum_{\nu=1}^{r}|L_\nu|\right)^r\left(\sum_{\nu=r+1}^{n+1}|L_\nu|\right)^{n+1-r}\leqq\frac{|\Delta|}{\rho^*_{n,r}},$$

$$|L_1\cdots L_{n+1}|\leqq\frac{|\Delta|}{\rho^*_{n,r}r^r(n+1-r)^{n+1-r}}."$$

On specializing the L's, results on the approximations of

$$|a_1x_1+\cdots+a_nx_n-x_{n+1}|$$

or of

$$\sum_{\nu=1}^{n}|a_\nu-x_\nu/x_{n+1}|$$

to zero are obtained. The proof makes use of the general method of H. F. Blichfeldt [Trans. Amer. Math. Soc. 15, 227–235 (1914)]; it suffices to consider the case that L_ν contains only $x_1, \cdots, x_\nu$. *K. Mahler* (Manchester).

Meulenbeld, B. Des approximations diophantiques d'un système de formes linéaires complexes. Nederl. Akad. Wetensch., Proc. 45, 924–928 (1942). [MF 10443]

The author states, without proof, results similar to those in the papers reviewed above, but now allows the forms to have complex coefficients. *K. Mahler* (Manchester).

ANALYSIS

*Levi, Beppo. **Systems of Analytic Equations: Equations in Finite Terms, Ordinary and Partial Differential Equations.** Monografias Publ. por la Facultad de Ci. Mat., Fisico-Quimicas y Nat., Apl. a la Industria. No. 1. Rosario, Argentina, 1944. 216 pp. (Spanish)

[The Spanish title is: Sistemas de ecuaciones analiticas en terminos finitos, diferenćiales y en derivadas parćiales. The phrase "in finite terms" means "not involving derivatives."] This monograph is a clearly written exposition of the fundamental existence theorems for systems of analytic partial differential equations, together with necessary preliminary material on implicit functions and ordinary differential equations. The first chapter is a rapid summary of the relevant parts of the theory of analytic functions, culminating in the Weierstrass preparation theorem and related results. The second chapter discusses first the solution of a system of k equations $f_m(x_1, \cdots, x_n)=0$ $(m=1, 2, \cdots, k; k<n)$, where the f_m are analytic, and applies the discussion to systems of analytic ordinary differential equations. Chapter 3 contains two algebraic lemmas on systems of monomials. Chapters 4 and 5 (about half the book) are devoted to systems of partial differential equations. After a preliminary discussion of equations of the first order, the author sets up a systematic process which after a finite number of steps must either show the system to be incompatible or reduce it to one whose form allows the calculation of the coefficients of its power series solutions. The possible convergence or divergence of the formal solutions is then discussed. The theory is illustrated by numerous interesting examples. *R. P. Boas, Jr.* (Cambridge, Mass.).

Mandelbrojt, S. **Some theorems connected with the theory of infinitely differentiable functions.** Duke Math. J. 11, 341–349 (1944). [MF 10672]

The first part of this paper contains remarkably simple constructions of functions showing that known conditions connected with Watson's problem and with quasi-analytic functions are best possible. Let $\{M_n^c\}$ denote the convex regularized sequence, regularized by logarithms, of the sequence $\{M_n\}$ with $M_n^{1/n}\rightarrow\infty$ [Mandelbrojt, Rice Inst. Pamphlet **29**, no. 1 (1942); these Rev. **3**, 292]. If

$$\sum M_n^c/M_{n+1}^c<\infty$$

and $\mu_1=2/M_1^c$, $\mu_n=2M_{n-1}^c/M_n^c$, $n>1$, the entire function

$$F(z)=\prod_{j=1}^{\infty}(1-e^{-\mu_j z})/(\mu_j z)$$

satisfies $0<|F(z)|<M_n/|z|^n$ in $x>0$, thus showing that the well-known result that $\sum M_n^c/M_{n+1}^c=\infty$ implies $F(z)\equiv 0$ if the same inequalities are satisfied cannot be improved. A construction almost as simple furnishes a function which is of class C^∞ in $[-\pi, \pi]$, not identically zero, vanishes with all derivatives as $\pm\pi$ and belongs to the class $C\{M_n\}$ if $\sum M_n^c/M_{n+1}^c<\infty$. In the second part of the paper a converse is given for a theorem of S. Bernstein on best approximation by polynomials on the whole real axis. Let $R(x)$ be positive, even and continuous. The order, with respect to $R(x)$, of approximation by polynomials to a given continuous $f(x)$ is specified by

$$E(R;f)=\lim_{n\to\infty}\ \liminf\ \sup_{-\infty<x<\infty}\frac{|f(x)-P_n(x)|}{R(x)},$$

where lim inf refers to all polynomials of degree n. Bernstein's theorem is that, for $R(x)$ of a certain form, with $\int^\infty x^{-2}\log R(x)dx$ divergent, $E(R;f)=0$ if $f(x)/R(x)\rightarrow 0$. The author now constructs a continuous $\varphi(x)$ with $\varphi(x)/R(x)\rightarrow 0$ and $E(R;f)>0$, provided that $\int^\infty x^{-2}\log R(x)dx$ converges and $\log R(x)$ is a convex function of $\log x$, $x>0$. *R. P. Boas, Jr.* (Cambridge, Mass.).

Calculus

*Johnson, Walter C. **Mathematical and Physical Principles of Engineering Analysis.** McGraw-Hill Book Co., New York, 1944. x+346 pp. $3.00.

The aim of this text is to present the essential physical and mathematical principles and methods of approach that underlie the solution of many practical engineering problems. As a background for the applications of the mathematical methods discussed in the text, the author develops some elementary principles in mechanics and electrodynamics. Throughout the book the balance between the methods of mathematics and physics is emphasized.

There are two excellent chapters on the graphical, numerical and analytical methods of solving ordinary differential equations. In addition to a discussion of ordinary differential equations with constant coefficients, such equations as Euler's and Bessel's equation are discussed. This chapter is followed by a treatment of complex quantities and their application to the steady state solution of ordinary linear nonhomogeneous differential equations which arise in electric circuit theory and mechanical vibration theory. The book closes with a discussion of Fourier series and the elements of partial differential equations. In addition to giving the method of separation of variables as a technic for solving partial differential equations, graphical methods are also discussed. The book abounds in many interesting problems. *A. E. Heins* (Cambridge, Mass.).

Neder, Ludwig. **Modell einer Differentialrechnung mit aktual unendlich kleinen Grössen erster Ordnung.** Math. Ann. 118, 251–262 (1941). [MF 10705]

This paper sets up a formal differential calculus by means of the introduction of "mixed variables" $[a, \alpha]$ which are essentially magnitudes consisting of a finite part a and an infinitesimal part α and for which rational, trigonometric and exponential operations are defined. For functions of these "mixed variables," whose functional values are also magnitudes of this kind, a process of differentiation is defined in such a way as to preserve the usual rules of the differential calculus, to within differentials. *A. Dresden.*

Neder, Ludwig. **Modell einer Leibnizischen Differentialrechnung mit aktual unendlich kleinen Grössen sämtlicher Ordnungen.** Math. Ann. 118, 718–732 (1943). [MF 10723]

Extending the concepts introduced in the paper reviewed above, the author now introduces "mixed variables" $[a_0, a_1, a_2, \cdots]$, in which a_0, a_1, etc. are real numbers, and which are essentially magnitudes consisting of a finite part a_0 and of infinitesimal parts a_1, a_2, etc., of successive orders. By means of a formal Taylor series, he obtains differentials of different orders for functions of these "mixed variables" whose functional values are of the same type. The dif-

ferential calculus which is thus set up reproduces the usual formulas of elementary calculus. As a formal exercise, the procedure of this paper is harmless and at times amusing. In the introductory paragraph the author says that the purpose of the paper is a demonstration of the noncontradictoriness of the differential calculus. As such, it can not be taken very seriously, since no system of undefined concepts and of unproved propositions is exhibited and because the formal Taylor series requires the derivatives of functions of a real variable in the ordinary sense. *A. Dresden.*

Uspensky, J. V. Elementary derivation of the series for $\sin x$ and $\cos x$. Math. Notae 4, 1–10 (1944). (Spanish) [MF 10696]

Taylor, A. E. Differentiation of Fourier series and integrals. Amer. Math. Monthly **51**, 19–25 (1944). [MF 10064]

Theorems and discussion, based on conditions of Dirichlet form, suitable for students unfamiliar with Lebesgue integrals. *R. P. Agnew* (Ithaca, N. Y.).

Roca, Marcelo Alonso. Generalizations of the theorems of Gauss and Stokes and definition of two integral operators. Revista Soc. Cubana Ci. Fís. Mat. 1, 109–114 (1943). (Spanish) [MF 10110]

The theorems of Gauss and Stokes if applied to tensors of any order lead to a symbolic relation between the corresponding integral operations. *I. Opatowski.*

Guarnieri, Angel J. On the integral $\int(1-x^2)^{-\frac{1}{2}}dx$. Revista Union Mat. Argentina **9**, 122–131 (1943). (Spanish) [MF 9854]

Dean, W. R. Note on the evaluation of an elliptic integral of the third kind. J. London Math. Soc. **18**, 130–132 (1943). [MF 10361]

The formula which reduces the Legendre complete elliptic integral of the third kind to integrals of first and second kind is proved by a new method. *I. Opatowski.*

Theory of Functions of Complex Variables

Pfluger, A. Über gewisse ganze Funktionen vom Exponentialtypus. Comment. Math. Helv. 16, 1–18 (1944).

Let $f(z)$ be an entire function of exponential type, $z_n = r_n e^{i\theta_n}$, $r_n \neq 0$, its zeros, $S(r) = \sum_{|z_n| \leq r} 1/z_n$. Under the assumptions (1) that the indicator diagram of $f(z)$ is a vertical line segment and (2) that

$$\lim_{R\to\infty} \int_1^R x^{-2} \log |f(x)f(-x)|\, dx$$

exists and is finite, it is known [Paley and Wiener, Cartwright, Levinson] that (a) the zeros have a positive density, (b) they have equal densities in the right and left half-planes, (c) $\sum |\sin \theta_n|/r_n$ converges. The author studies the relationships among the hypotheses (1), (2), and the conclusions (a), (b), (c) separately. He finds that (a) and (c) together are equivalent to (2); a necessary and sufficient condition that (1) should also be true is that (d) $\lim_{r\to\infty} S(r)$ exists; (d) is not in general implied by (a), (b) and (c). The author obtains additional, more detailed information about the relationships among the conditions and the indicator diagram. *R. P. Boas, Jr.* (Cambridge, Mass.).

Pfluger, A. Über Interpolation ganzer Funktionen. Comment. Math. Helv. 14, 314–349 (1942).

The author considers the problem of extending to a general class of sequences $\{a_n\}$ the theorem of himself and Ganapathy Iyer that an entire function of order 2 and type less than $\pi/2$, bounded at the lattice points, must be a constant. He makes use of his earlier results on the growth and distribution of values of entire functions of nonintegral order [Comment. Math. Helv. 11, 180–214 (1938); 12, 25–65 (1939); these Rev. 1, 113], together with extensions to functions of integral order and mean type. The results of this paper contain those of B. J. Maitland [Proc. London Math. Soc. (2) **45**, 440–457 (1939); these Rev. **1**, 49] and B. Lévine [Rec. Math. [Mat. Sbornik] N.S. **8(50)**, 437–454 (1940); these Rev. **2**, 182] and represent an especial improvement for functions of integral order, in that the sequences $\{a_n\}$ considered are not required to satisfy any strong conditions of symmetry. The author observes that it is enough to obtain conditions on the a_n which will cause the Lagrange interpolation formula

$$\pi(z) \sum_{n=1}^{\infty} \frac{G(a_n)}{\pi'(a_n)(z-a_n)}$$

to represent all entire functions $G(z)$ of specified growth, where $\pi(z)$ is the canonical product over the a_n. He studies the interpolation problem in two steps, by first obtaining sufficient conditions on $\pi(z)$ in terms of the Phragmén-Lindelöf indicator function for $G(z)$ with respect to a given approximate order, and then obtaining conditions on the a_n which will make $\pi(z)$ satisfy the conditions in question. The conditions on $\{a_n\}$ include a kind of measurability and a condition which keeps the a_n relatively uncondensed. Only a simple special case can be quoted here, namely that, for order 2, type less than $\pi/2$, it is sufficient that the a_n should be obtained by displacing each lattice point $m+in$ by an amount not exceeding a fixed S, provided that $\liminf_{j\neq k} |a_j - a_k| > 0$ when the a_k are arranged in order of increasing absolute value. *R. P. Boas, Jr.*

Kober, H. Approximation by integral functions in the complex domain. Trans. Amer. Math. Soc. 56, 7–31 (1944). [MF 10817]

In an earlier paper [Trans. Amer. Math. Soc. **54**, 70–82 (1943); these Rev. 4, 271] the author has discussed approximation by entire functions to functions defined on $-\infty < x < \infty$ or $0 < x < \infty$. He now considers the same problem for functions defined in an angle of opening Θ, $0 < \Theta < 2\pi$, or in a strip. Typical result: if $F(z)$ is analytic, not constant, and bounded in the open angle, and satisfies an auxiliary continuity condition in the closed angle, then it can be uniformly approximated by entire functions of order $\pi/(2\pi-\Theta)$, but not by entire functions of lower order. Sharper results for half-planes ($\Theta = \pi$) deal also with L^p approximation to functions of class $\mathfrak{H}_p$. Unbounded functions, in particular z^r, are also discussed. For the strip, the appropriate approximating functions are of order one and finite type; approximation by rational functions is also discussed. The discussion for the strip is based on a factorization theorem, representation theorems, etc., analogous to those involved in the Hille-Tamarkin theory of classes

$\mathfrak{H}_p$ in a half-plane, and of independent interest in themselves. *R. P. Boas, Jr.* (Cambridge, Mass.).

Piranian, George. Algebraic-logarithmic singularities and Hadamard's determinants. Duke Math. J. 11, 147–153 (1944). [MF 10155]

Let $s=u+iv$, let k be a nonnegative integer and let $\phi(z)$ be a function regular at z_0, with $\phi(z_0)\neq 0$. The expression

$$g(z)=e^{-s\log(z-z_0)}[\log(z-z_0)]^k\phi(z),$$

with $\log(z-z_0)$ being real for $z-z_0$ positive, is said to be an algebraic-logarithmically singular element at z_0. The element is said to have compound order (u, k) if $s\neq 0, -1, -2, \cdots$, order $(u, k-1)$ if $s=0, -1, -2, \cdots, k>0$, order $(-\infty, 0)$ if $s=0, -1, -2, \cdots, k=0$. The following theorem is proved. If $f(z)=\sum(a_n/z^n)$ is holomorphic in the region $|z|>|z_0|>0$ except for poles $z_1, z_2, \cdots, z_m$ of multiplicities $r_1, r_2, \cdots, r_m$, with $\sum_{i=1}^m r_i=P$, and if the only singularity of $f(z)$ on the circle $|z|=|z_0|$ is an ordinary algebraic-logarithmic singularity of compound order (u, k) at z_0, then

$$u=1+\lim_{n\to\infty}\frac{\log|D_{n,p}|-n\log\left(|z_0|\prod_1^m|z_i|^{r_i}\right)}{\log n},$$

$$k=\lim_{n\to\infty}\frac{\log|D_{n,p}|-n\log\left(|z_0|\prod_1^m|z_i|^{r_i}\right)-(u-1)\log n}{\log\log n},$$

where

$$D_{n,p}=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+p} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+p+1} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+p} & a_{n+p+1} & \cdots & a_{n+2p} \end{vmatrix}$$

(Hadamard's determinant). [For the definition of algebraic-logarithmic singularities, see ~~Jungen~~, Comment. Math. Helv. 3, 266–306 (1931).] Jungen *S. Mandelbrojt.*

Heins, Maurice H. On a problem of Walsh concerning the Hadamard three circles theorem. Trans. Amer. Math. Soc. 55, 349–372 (1944). [MF 10497]

The author considers the class of functions $f(z)$ regular in $|z|<1$ and with $|f(z)|<1$ there, and also satisfying the inequality $|f(z)|\leqq m$ for $|z|=r$. The maximum possible value of $|f(z)|$ for $|z|=\rho$ is sought $(r<\rho<1)$. The principal result is that this maximum is attained for a certain rational function $f_0(z)$ with $|f_0(z)|=1$ for $|z|=1$, and that the degree of $f_0(z)$ is n when $r^n\leqq m<r^{n-1}$. If $m=r^n$, then $f_0(z)=z^n$, and $|f_0(z)|=m$ for $|z|=r$; but, if $r^n<m<r^{n-1}$, the equality $|f_0(z)|=m$ holds at only n points on $|z|=r$. Essential in the proof is the theory of interpolation by bounded functions, the points just mentioned being used as interpolating points. The corresponding problem when $f(z)$ is not supposed regular for $|z|<r$ was first solved by O. Teichmüller [Deutsche Math. 4, 16–22 (1939)]; a different method was used by the reviewer [Duke Math. J. 10, 341–354 (1943); these Rev. 4, 241]. [See also the reviewer's address [to appear in Bull. Amer. Math. Soc.], where both problems are discussed.] A further problem treated by the author concerns functions with $|f(z)|<1$ for $|z|<1$ and with $|f(z)|\leqq m$ at a certain set of points on the real axis; the extremal functions are of the same type as before. *R. M. Robinson* (Berkeley, Calif.).

Robinson, Raphael M. Bounded analytic functions. Univ. California Publ. Math. (N.S.) 1, 131–146 (1944). [MF 10558]

This paper is concerned with the following problems in the theory of bounded analytic functions. (I) Let $g(z)$ denote a function which is analytic and of modulus not greater than one for $|z|<1$. Let β denote a point of the interior of the unit circle different from zero, and let $\alpha=g(0)$, $\mu=g(\beta)$, $\nu=g'(\beta)$. It is proposed to determine the inequalities relating α, β, μ, ν as well as the inequalities relating $|\alpha|, |\beta|, |\mu|, |\nu|$. All possibilities are considered, the admissible relations for all subsets of $(\alpha, \beta, \mu, \nu)$ and $(|\alpha|, |\beta|, |\mu|, |\nu|)$ being taken into account. The proofs depend upon Schwarz's lemma, but much of the analytic work consists in the elimination of various parameters. The results obtained are sharp and the treatment is exhaustive. (II) Let $f(z)$ denote a function analytic and of modulus less than one for $|z|<1$ and satisfying $f(0)=0$. Here again β is a point of the interior of the unit circle different from zero and now $\alpha=f'(0)$, $\gamma=f(\beta)$, $\delta=f'(\beta)$. The problem is to determine the inequalities relating $\alpha, \beta, \gamma, \delta$ and the inequalities relating $|\alpha|, |\beta|, |\gamma|, |\delta|$. This problem is treated in the same spirit as (I). The results obtained are sharp and all cases are considered. *M. H. Heins.*

Wolff, J. Deux théorèmes sur la dérivée d'une fonction holomorphe univalente et bornée dans un demi-plan au voisinage de la frontière. Nederl. Akad. Wetensch., Proc. 45, 574–577 (1942). [MF 10423]

Let $f(z)$ denote a function which is analytic, univalent and bounded in the right-half plane $\Re z>0$. Let S_0 denote a sequence of points $z_n=x_n+iy_n$ $(n=1, 2, \cdots)$ of $\Re z>0$ such that $x_{n+1}<cx_n$ for $n=1, 2, \cdots$, where c is a positive constant less than one. For each real value of t let S_t denote the sequence $\{z_n+it\}$. It is shown that, on the sequence S_t,

$$\lim_{n\to\infty}f'(z_n+it)\sqrt{x_n}=0$$

for almost all t. The proof is based upon Koebe's distortion theorem and the fact that the area of the image of $\Re z>0$ with respect to $f(z)$ is finite. It is further shown that to every real p between 0 and $\frac{1}{2}$ there correspond functions $f(z)$ which are analytic, univalent and bounded for z in $\Re z>0$ such that, on each curve $y=x^p+t$ $(-\infty<t<+\infty)$,

$$\varlimsup_{z\to it}|f'(z)|\cdot\sqrt{x}=+\infty.$$

This is done by the construction of examples of functions having the desired properties. *M. H. Heins.*

Basilewitsch, J. D'une propriété extrémale de la fonction $F(z)=z/(1-z)^2$. Rec. Math. [Mat. Sbornik] N.S. 12(54), 315–319 (1943). (Russian. French summary) [MF 10225]

Using Löwner's method, the author proves the following result. Let $f(z)=z+c_2z^2+c_3z^3+\cdots$ be regular and schlicht in $|z|<1$ with all its coefficients c_r real. Then

$$2\arctan\left(\tan\frac{\phi}{2}\cdot\frac{1-r}{1+r}\right)\leqq\arg f(z)\leqq 2\arctan\left(\tan\frac{\phi}{2}\cdot\frac{1+r}{1-r}\right),$$

where $\phi=\arg z$, $r=|z|$. There is equality on the left only for $f=z/(1+z)^2$, on the right only for $f=z/(1-z)^2$. *D. C. Spencer* (New York, N. Y.).

LaGuardia, Rafael. Conformal mapping of a domain onto a circle. Bol. Fac. Ingen. Montevideo 2 (Año 7), 161–190 (1942). (Spanish) [MF 10839]

This is an expository paper designed particularly for engineering students who make use of certain results of the theory of conformal mapping without having made an overall study of the existence theorems connected with this topic. The ideas pertaining to these existence theorems have been collected here to fill this gap. In the first chapter the general notions of metric spaces, convergence, normal families, compact families and the like are drawn up in general abstract form for greater clarity of the connection of the ideas. In the second chapter the proofs of the fundamental existence theorems are given, using the modern methods of analytic functions. *M. S. Robertson.*

Rosenblatt, Alfred. On Mr. L. Kantorovich's method in the theory of conformal mapping and on the application of that method to aerodynamics. Actas Acad. Ci. Lima 6, 199–219 (1943). (Spanish) [MF 10762]

The method of L. Kantorovich [Rec. Math. Moscow **40**, 294–325 (1933)] for the practical determination of the conformal representation of plane domains on the interior of the unit circle is studied and a simple application is made. *E. F. Beckenbach* (Austin, Tex.).

Rosenblatt, Alfred. Some applications of Kantorovich's method of conformal mapping of plane domains to aerodynamics. Actas Acad. Ci. Lima 6, 236–249 (1943). (Spanish) [MF 10763]

The author continues his work on the method of L. Kantorovich [see the preceding review], now considering the maps of certain pisciform profiles on the unit circle. *E. F. Beckenbach* (Austin, Tex.).

Neumann, Ernst Richard. Inversion und konforme Abbildung von Komplementärgebieten. Math. Ann. 118, 276–285 (1941). [MF 10707]

In two earlier papers [Math. Ann. **116**, 534–554, 664–700 (1939)] the author has solved the problem of determining the mapping function for either of the two regions bounded by a sufficiently smooth Jordan curve in terms of the mapping function for the other. This was done by showing that either mapping function determines and is determined by a pair of distributions, the "natürlichen Belegung" and the "Grundrestbelegung," on the curve itself. In addition, he showed that the latter distribution is unnecessary in passing from the mapping function for the interior region to that for the exterior. In the present paper he shows by using inversion that it can therefore also be eliminated in passing from the exterior region to the interior. *L. H. Loomis* (Cambridge, Mass.).

Bolder, H. Sur le théorème de déformation de Koebe. Nederl. Akad. Wetensch., Proc. 45, 553–558 (1942). [MF 10420]

Let Ω denote a simply-connected region in the finite w-plane containing $w=0$ in its interior, for which Green's function exists. Let $G(0, w)$ denote the Green's function of Ω with pole at $w=0$ and $g(w)=G(0, w)-\log(1/|w|)$ for $w\neq 0$, defined at $w=0$ to be continuous there. Furthermore, d denotes the distance from $w=0$ to the boundary of Ω. Let Ω^* denote a Schlitzgebiet containing $w=0$ in its interior for which d is the distance from $w=0$ to the boundary of Ω^*. Let G^* and g^* have the meanings for Ω^* that G and g have for Ω. Then $g(0)\leqq g^*(0)$, equality occurring only if Ω is also a Schlitzgebiet. This theorem is demonstrated by purely potential-theoretic methods without reference to the theory of conformal mapping. *M. H. Heins*

Bolder, H. Sur une démonstration simple du théorème de déformation de Koebe, et d'un théorème du type Carleman-Milloux. Nederl. Akad. Wetensch., Proc. 45, 833–835 (1942). [MF 10438]

The theorem of the note reviewed above is extended. It is now assumed that Ω is a region (not necessarily simply-connected) in the finite w-plane containing $w=0$, for which the Green's function exists. Let $G(0, w)$ denote the Green's function of Ω with pole at $w=0$, for w in the closure of Ω, and be defined to be equal to zero exterior to Ω. Otherwise the notation of the above note is preserved. It is shown by potential-theoretic methods employed by Brelot [Bull. Soc. Roy. Sci. Liége **8**, 385–391 (1939)] in similar connections that $g(0)\leqq g^*(0)$, provided that there exists a point t of the boundary of Ω for which $|t|=d$ such that every circle $|w-t|=r$ contains at least one point w for which $G(0, w)=0$. Related questions are considered. *M. H. Heins* (Washington, D. C.).

Monna, A. F. Sur les fonctions univalentes. Nederl. Akad. Wetensch., Proc. 45, 826–832 (1942). [MF 10437]

Let $F(z)$ denote a function analytic and univalent for $|z|<1$ which admits a representation as the quotient of functions which are bounded and analytic for $|z|<1$. Let $F(0)=0$ and $F(z)\neq w_0$ for $|z|<1$. The radial limit $\lim_{r\to 1}F(re^{i\theta})$ is known to exist for almost all θ. It is denoted by $F(e^{i\theta})$. Let E_R denote the subset of $[0\leqq\theta<2\pi]$ for which $|F(e^{i\theta})-w_0|\geqq R$, and let mE_R denote the measure of E_R. Furthermore, let $C_{t,\rho}$ denote the curve defined in $|z|\leqq\rho$ $(0<\rho<1)$ by $|F(z)-w_0|=t$ and let

$$l(t)=\lim_{\rho\to 1}\int_{C_{t,\rho}}|F'(z)|\,ds.$$

It is shown that, for $R\geqq|w_0|$,

$$mE_R\leqq 2\pi\exp\left[-\pi\int_{|w_0|}^{R}dt/l(t)\right].$$

This inequality is applied to problems in the theory of the conformal mapping of convex domains. *M. H. Heins.*

Monna, A. F. Sur quelques inégalités de la théorie des fonctions et leurs généralisations spatiales. I. Nederl. Akad. Wetensch., Proc. 45, 43–50 (1942). [MF 10377]

Monna, A. F. Sur quelques inégalités de la théorie des fonctions et leurs généralisations spatiales. II. Nederl. Akad. Wetensch., Proc. 45, 165–168 (1942). [MF 10383]

These papers consist in a collection of remarks on Koebe's and related inequalities for complex functions and the possibility of proving analogous inequalities in three dimensions. If the Green's function $G(P, P_0)$ for a two dimensional region is written in the form $G(P, P_0)=\log(1/PP_0)-g(P, P_0)$, then Koebe's inequality amounts to $g(P_0, P_0)\geqq\log(1/4d)$, where d is the minimum distance of P_0 to points of the boundary of the region. The author obtains a proof, by using properties of the harmonic measure, of an inequality of the form $g(P_0, P_0)\geqq\log(K/d)$, where, however, K is less than $\frac{1}{4}$, the virtue in the proof being that conformal mapping was not directly used. The analogous inequality in three dimensions $g(P_0, P_0)\geqq K/d$ does not hold, except for the trivial case $K=0$, as the author points

out. It does hold, however, if the domains Ω considered are restricted to those which lie inside a domain Ω^* with the following properties: ∞ is not an interior point of Ω^*, and the portion of the boundary of Ω^* contained in any open set having at least one point in common with the boundary has positive harmonic measure with respect to a point of Ω^*. In this case, the author proves the existence of a constant K depending on Ω^* such that $K(\Omega^*)/d \leqq g(P_0, P_0)$. Certain nonoptimum bounds are also established for the size of the level surfaces of the Green's function in three dimensions and the special case when Ω^* consists of the whole space less a half plane is worked out in detail. *J. W. Green.*

Maass, Hans. Theorie der Poincaréschen Reihen zu den hyperbolischen Fixpunktsystemem der Hilbertschen Modulgruppe. Math. Ann. **118**, 518–543 (1942). [MF 10713]

Recently the theory of automorphic functions of a single variable has been greatly simplified by the application of an integral covariant, due to H. Petersson; in particular, Petersson has proved that the set of automorphic forms of given real dimension (<-2), with a multiplier system of absolute value 1 for a group of first kind, possesses a finite basis consisting of Poincaré series of special simple types [Abh. Math. Sem. Hansischen Univ. **14**, 22–60 (1941); these Rev. **3**, 204]. In a former publication the author generalized Petersson's investigation by considering certain classes of automorphic functions of several variables, in particular for the case of Hilbert's modular group [Math. Ann. **117**, 538–578 (1940); these Rev. **2**, 87]; however, he introduced only the Poincaré series corresponding to the first of Petersson's three types (parabolic fixed points). In the present paper his results are extended to the two other types of Poincaré series. The proof used Dirichlet's theorem concerning algebraic units. *C. L. Siegel.*

Nef, Walter. Über die singulären Gebilde der regulären Funktionen einer Quaternionenvariabeln. Comment. Math. Helv. **15**, 144–174 (1943).

Der Verfasser untersucht die Darstellung einer rechtsregulären Funktion einer Quaternionenvariablen, die im Raum R_4 als einziges (unwesentliches oder wesentliches) singuläres Gebilde eine geradlinige Strecke besitzt. Eine solche Funktion lässt sich nach der Verfasser im ganzen R_4 mit Ausnahme des singulären Gebildes durch gewisse unendliche Reihen darstellen, deren Glieder Stieltjèssche Integrale sind. Nimmt man der Einfachheit halber als singuläres Gebilde die Strecke $-1 \leqq c_0 \leqq +1$ der reellen Achse an, so haben die Integrale die Form:

$$\int_{-1}^{+1} d[\delta(c_0)]\cdot q(z-c_0).$$

Hierbei sind die q gewisse einfache linksreguläre Funktionen und die δ Funktionen von c_0 mit in $-1 \leqq c_0 \leqq +1$ beschränkter Schwankung. *P. Thullen* (Quito).

Fueter, Rud. Die Funktionentheorie der Dirac'schen Differentialgleichungen. Comment. Math. Helv. **16**, 19–28 (1944).

The author applies his theory of regular functions of a hypercomplex variable to the solution of the Dirac equations in the case of vanishing restmass. He first develops two interrelated systems of hypercomplex numbers L_c and A_e with units c_k and e_k, respectively, where $k=0, 1, 2, 3$. The numbers of A_e form an algebra, while those of L_c, which are the Clifford numbers, form only a linear system. Both systems are associative. He then considers functions $W(z)$ and $w(z)$, called c-functions and e-functions, which are defined over a region H of L_c in both cases and take values in L_c and A_e, respectively. The notions of regular c-function and left-regular e-function are then defined. It turns out that an e-function is left-regular if its components satisfy the Dirac equations.

It is shown that a necessary and sufficient condition that an e-function w be left-regular in the region H (and hence have its components satisfy the Dirac equations there) is that, for every regular and nonvanishing c-function W, a certain surface integral $\int_R WwdZ$, which the author defines, should vanish for every closed orientable hypersurface R of H. A linear integral equation is derived which gives the value of a left-regular e-function at any point in terms of its values on a given surface and on a given hypercone. It is brought out that in this theory the Dirac equations play the role of the Cauchy-Riemann equations in the theory of functions of an ordinary complex variable. *O. Frink.*

Fourier Series and Generalizations, Integral Transforms

***Hardy, G. H. and Rogosinski, W. W. Fourier Series.** Cambridge Tracts in Mathematics and Mathematical Physics, no. 38. Cambridge University Press, 1944. 100 pp. $1.75.

This is a modern treatment of Fourier series, concise, "yet full enough to serve as an introduction to Zygmund's standard treatise." The knowledge assumed is roughly chapters X–XII of Titchmarsh's "Theory of Functions." While self-contained, some preliminary acquaintance with the subject is advisable. The whole treatment is theoretical, there being no discussion of applications to boundary-value or other physical problems.

The following is a brief summary of the contents by chapter. (I) Generalities. Preliminary definitions and a discussion of the L^p spaces. (II) Fourier series in Hilbert space. Here are discussed general orthogonal expansions in L^2 and the elementary properties of the trigonometric system: completeness, the Parseval and Riesz-Fischer theorems. (III) Further properties of trigonometrical Fourier series. The Riemann-Lebesgue theorem is established in a more general form than usual. Included are theorems on the order of magnitude of the coefficients, a proof of the well-known theorem on term-by-term integration, a treatment of series with decreasing coefficients and an introduction to the Gibbs phenomenon. (IV) Convergence of Fourier series. The usual tests for convergence are established, together with analogous tests for the conjugate series. Lebesgue's test is given in a general form due to Gergen. (V) Summability of Fourier series. The chapter begins with a discussion of general methods of summability. A novel feature is the treatment of the so-called K methods in §§ 5.5 and 5.6. Particular emphasis is laid on the $(C, 1)$ and A methods for the series and its conjugate. There is no material on Cesàro summation of general order in the text proper, but a note at the end of the book [p. 97] sketches the results briefly and provides further references. (VI) Applications of the theorems of chapter V. The chief results established: there are Fourier series which diverge almost everywhere; if $f \epsilon L^2$ and $n_{\nu+1}/n_\nu \geqq \lambda > 1$, then $s_{n_\nu} \to f$ almost everywhere. There is a proof of the existence of the conjugate function

if $f \epsilon L$, and the chapter concludes with the theorem of Kuttner to the effect that, if a Fourier series converges on a set E of positive measure, its conjugate series converges almost everywhere in E. (VII) General trigonometrical series. This includes the Riemann theory of Fourier series and culminates in the important theorem 100: if a trigonometrical series converges, except on a denumerable set E, to a finite integrable function f, then it is the Fourier series of f.

A word about omissions. Not included are the inequalities of Young and Hausdorff, the Paley theory, or results which depend on complex variable theory. However, an ample set of notes at the end surveys omitted topics and provides the interested reader with references. *H. Pollard.*

Chen, Kien-Kwong. On the absolute Cesàro summability of negative order for a Fourier series at a given point. Amer. J. Math. 66, 299–312 (1944). [MF 10574]

It is known that, if $f(x)$ is of period 2π, of bounded variation in $(0, 2\pi)$ and belongs to Lip k, $k>0$, then the Fourier series of $f(x)$ is absolutely convergent [see Zygmund, J. London Math. Soc. 3, 194–196 (1928)]. The author shows that under the above conditions the Fourier series of f is summable $|C, -\frac{1}{2}k+\epsilon|$, $\epsilon>0$. Extensions to negative summability of some other known results concerning absolute convergence of Fourier series are given. *A. Zygmund.*

Bayard, Marcel. Sur la représentation des fonctions d'une variable réelle en séries trigonométriques plus générales que les séries de Fourier. C. R. Acad. Sci. Paris 216, 792–793 (1943). [MF 10654]

The paper indicates that under certain conditions the classical development of a function in a series of exponentials [see Picard, Traité d'Analyse, Gauthier-Villars, Paris, 1926, vol. 2, 3rd ed., p. 187] can be made with preassigned phases, in a reduced interval. *R. Salem.*

Sirvint, G. Quelques exemples de séries de Dirichlet dont la suite d'exposants est condensée. Rec. Math. [Mat. Sbornik] N.S. 12(54), 370–376 (1943). (French. Russian summary) [MF 10230]

The author gives two examples of Dirichlet series with the maximum density of the exponents equal to infinity and such that they overconverge in a larger half-plane than their half-plane of convergence and which are holomorphic in a larger half-plane than their half-plane of overconvergence. *S. Mandelbrojt* (Houston, Tex.).

Alexits, Georges. Sur la convergence des séries de polynomes orthogonaux. Comment. Math. Helv. 16, 200–208 (1944).

The author proves among others the following theorem. Let $0 \leqq w(x) < W$ be a weight function, $p_n(x)$ the sequence of orthogonal polynomials,

$$\int_{-1}^{+1} p_n(x)p_m(x)w(x)dx=0, \quad \int_{-1}^{1} w(x)p_n^2(x)dx=1.$$

Assume $|p_n(x)| \leqq P$ for $-1 \leqq x \leqq 1$ and all n. Then, if $\sum^\infty c_n^2 \log n$ converges, $\sum_{n=0}^\infty c_n p_n(x)$ converges almost everywhere. He also obtains results about the absolute convergence of $\sum_{n=0}^\infty c_n p_n(x)$. *P. Erdös* (Lafayette, Ind.).

Wall, H. S. Continued fractions and bounded analytic functions. Bull. Amer. Math. Soc. 50, 110–119 (1944). [MF 9895]

The author gives a new proof of his theorem [Trans. Amer. Math. Soc. 48, 165–184 (1940); these Rev. 2, 90] that the sequence $\{c_p\}$ of real numbers, of which $c_0=1$, is totally monotone if and only if the power series $\sum(-1)^p c_p w^p$ has an expansion as a continued fraction of the form $1/[1+K(a_n w/1)]$, where $a_1=g_1$, $a_n=(1-g_{n-1})g_n$ $(n=2, 3, \cdots)$, $0 \leqq g_p \leqq 1$ $(p=1, 2, \cdots)$. The proof is based on well-known theorems of Schur and of Riesz which characterize, respectively, holomorphic functions bounded in the unit circle and holomorphic functions which assume the value unity for $z=0$ and which have nonnegative real parts in the unit circle. Proofs are given for each of these theorems; in the case of Schur's theorem the author makes use of continued fraction analysis and avoids Schur's pseudo-continued fraction. *W. Leighton* (New York, N. Y.).

Widder, D. V. The iterates of the Laplace kernel. Duke Math. J. 11, 231–250 (1944). [MF 10663]

The iterates of the Laplace kernel are defined by

$$G_n(x, y)=\int_0^\infty G_0(x, t)G_{n-1}(t, y)dt, \quad G_0(x, y)=e^{-xy}.$$

For odd n they have been calculated by the author and are rational functions of x, y and $\log(x/y)$ [Bull. Amer. Math. Soc. 43, 813–817 (1937)]. It is now shown that the even iterates are expressible in terms of the transcendentals

$$\int_0^\infty e^{-xt}((\log t)^p/(1+t))dt, \quad p=0, 1, 2, \cdots,$$

$$\int_0^\infty e^{-xt}((\log t)^p/(t-1))dt, \quad p=1, 2, 3, \cdots.$$

For the sake of future investigations of the iterates of the Laplace transform, the author investigates the asymptotic behavior of the $G_n(x, y)$ as $x\to\infty$ and as $x\to 0+$, obtaining an asymptotic series about ∞ and a convergent series about 0. The dominant terms are as follows: as $x\to\infty$,

$$G_{2n}(x, y)\sim\frac{(\log x)^{n-1}}{(n-1)!xy}, \quad G_{2n-1}(x, y)\sim\frac{(\log x)^{n-1}}{(n-1)!x};$$

as $x\to 0+$,

$$G_{2n}(x, y)\sim\frac{(\log(1/x))^n}{n!}, \quad G_{2n-1}(x, y)\sim\frac{(\log(1/x))^{n-1}}{(n-1)!y}.$$

R. P. Boas, Jr. (Cambridge, Mass.).

Pollard, Harry. An inversion formula for the Stieltjes transform. Duke Math. J. 11, 301–318 (1944). [MF 10669]

An inversion is given for the integrals

$$(1)\quad f(x)=\int_0^\infty \frac{\varphi(t)dt}{x+t}, \qquad (2)\quad f(x)=\int_0^\infty \frac{d\alpha(t)}{x+t}$$

in terms of the values of $f(x)$ on the real axis. For (1) it reads

$$\varphi(x)=\lim_{k\to\infty}\frac{(-1)^k(2k-1)!x^{k-1}}{k!(k-2)!}\int_0^\infty u^k f(u)\frac{\partial^{2k-1}}{\partial u^{2k-1}}\left\{\frac{u^{2k-1}}{(u+x)^{2k}}\right\}du$$

for almost all x. Necessary and sufficient conditions, involving the inversion operator, are given for a function to have the representations (1), (2) with $\varphi(t)$ and $\alpha(t)$ in various classes, including the case where $\alpha(t)$ is of bounded variation in $(0, R)$ for each positive R. For example, $f(x)$ has the representation (2) with nondecreasing $\alpha(t)$ if and only if $f(x)$ is continuous for $x>0$, $f(\infty)=0$, and the expression under the sign "lim" in the inversion formula quoted above is nonnegative for $x>0$ and $k=2, 3, \cdots$. *R. P. Boas, Jr.*

Reed, I. S. On the solution of a general transform. Proc. Nat. Acad. Sci. U. S. A. **30**, 169–172 (1944). [MF 10796]

Goodspeed, generalizing a formula of Ramanujan, has given a class of series, involving entire functions, which are Watson transforms of each other with an appropriate kernel [Quart. J. Math., Oxford Ser. **10**, 210–218 (1939); these Rev. **1**, 74]. The author extends this work to unsymmetrical Watson transforms and illustrates the result with an example. *R. P. Boas, Jr.* (Cambridge, Mass.).

Guinand, A. P. Functional equations and self-reciprocal functions connected with Lambert series. Quart. J. Math., Oxford Ser. **15**, 11–23 (1944). [MF 10684]

The Lambert series considered is

$$\varphi(z)=\sum_{n=1}^{\infty}\frac{1}{e^{2\pi nz}-1}=\sum_{n=1}^{\infty}d(n)e^{-2\pi nz}.$$

Writing $F(z)=\varphi(z)-(2\pi z)^{-1}(\gamma-\log 2\pi z)$, $\Re(z)>0$, the author shows that $F(z)$ is self-reciprocal with respect to the Fourier kernel $2x\pi^{-1}(x^2-1)^{-1}$ and that $F(z)+(i/z)F(1/z)$ can be continued analytically over the z-plane cut from 0 to $+i\infty$, while $F(z)-(i/z)F(1/z)$ can be continued over the z-plane cut from 0 to $-i\infty$. If square brackets denote the analytic continuation of the whole expression inside them, then $F(z)$ satisfies the functional equation

$$[F(z)-(i/z)F(1/z)]+[F(-z)-(i/z)F(-1/z)]=0$$

except for z on a cut from 0 to $-i\infty$; $\varphi(iy)$, for y not on the negative real axis, satisfies

$$\left[\varphi(iy)-\frac{1}{y}\varphi\left(-\frac{i}{y}\right)\right]+\left[\varphi(-iy)-\frac{1}{y}\varphi\left(\frac{i}{y}\right)\right]=\frac{1}{2}-\frac{1}{2y};$$

and, for real positive y,

$$\lim_{\alpha\to+0}\left\{\sum_{n=1}^{\infty}d(n)e^{-2\pi ny\sin\alpha}\cos(2\pi ny\cos\alpha)\right.$$
$$\left.-(1/y)\sum_{n=1}^{\infty}d(n)e^{-2\pi n(\sin\alpha)/y}\cos(2\pi n(\cos\alpha)/y+\alpha)\right\}$$
$$=\tfrac{1}{4}-1/4y.$$

The last relation gives a meaning to the formal result, obtainable from Voronoï's summation formula,

$$-\tfrac{1}{4}+\sum_{n=1}^{\infty}d(n)\cos 2\pi ny$$
$$=-(1/4y)+(1/y)\sum_{n=1}^{\infty}d(n)\cos(2\pi n/y),$$

in which the series neither converge nor are summable by any ordinary method. Similar results are given for series involving $\sigma_k(n)$, the sum of the kth powers of the divisors of n. *R. P. Boas, Jr.* (Cambridge, Mass.).

Polynomials, Polynomial Approximations

Turrittin, H. L. Asymptotic distribution of zeros for certain exponential sums. Amer. J. Math. **66**, 199–228 (1944). [MF 10568]

The author gives asymptotic approximations for the number and location of the zeros of

$$E(z)=\sum_{j=1}^{J}P_j(z)\exp q_j(z),$$

where P and q represent polynomials of the complex variable z. He shows that all the zeros lie in a finite number of regions, one of which is a fixed circle C_0, while the others (called "0-bands") go to infinity between certain bounding curves which approach certain lines asymptotically. He then shows that corresponding to each 0-band there exists a polynomial $P(r)$ such that, for a relatively dense set of positive values $r=r_1, r_2, \cdots$, we have $N(r)=P(r)+O(1)$, where $N(r)$ is the number of zeros in the 0-band outside of C_0 and inside the circle $|z|<r$. *R. H. Cameron.*

Lipka, Stephan. Integralsätze über Polynome mit lauter reellen Nullstellen. Math. Ann. **118**, 485–496 (1942). [MF 10711]

Some inequalities are proved concerning the value of the integral of a polynomial

$$f(x)=(x-x_1)(x-x_2)\cdots(x-x_n)$$

between two successive zeros of its first derivative

$$f'(x)=n(x-y_1)(x-y_2)\cdots(x-y_{n-1}),$$

where the x_j are all real and where $x_1\leqq x_2\leqq\cdots\leqq x_n$ and $y_1\leqq y_2\leqq\cdots\leqq y_{n-1}$. Specifically, it is proved by induction on n that, if $x_2-x_1\geqq x_3-x_2$ and if $f(x)>0$ for $x_1<x<x_2$, then $\int_{y_1}^{y_2}f(x)dx\geqq 0$, with the equality sign holding only when either $n=3$, $x_2-x_1=x_3-x_2$, or $x_1=x_2=x_3$. From this theorem applied to $f'(x)$, it follows that, if $y_2-y_1\geqq y_3-y_2$, then the two smallest zeros of $f''(x)$ are either both in the interval (x_2, x_3) or both outside this interval. Finally, it is proved that, if $x_{n-4}\leqq 0$ and $x_{n-2}=1$ and if

$$x_{n-3}\geqq 2\left[\int_{x_{n-2}}^{2}g(x)xdx\Big/\int_{x_{n-2}}^{2}g(x)dx\right],$$

where

$$g(x)=(x-x_1)(x-x_2)\cdots(x_n-x_{n-3}),$$

then $\int_{x_{n-4}}^{2}f(x)dx>0$. *M. Marden* (Milwaukee, Wis.).

Lipka, Stephan. Über die Irreduzibilität von Polynomen. Math. Ann. **118**, 235–245 (1941). [MF 10703]

Various sufficient conditions are given for a polynomial $f(x)=x^n+a_1x^{n-1}+\cdots+a_n$, with integral coefficients a_j, to be irreducible in the field of rational numbers. It is proved that, if any one of the following sets of conditions is satisfied, $f(x)$ will be irreducible.

(1) $|a_1|>|1+a_2|+|a_3|+\cdots+|a_n|$.

(2) $a_2>0$, $\sqrt{a_2}>A(|a_1|+|a_3|+\cdots+|a_n|)$, $A=3/\sqrt{2}$.

(3) $a_1=0$, $a_2>0$, $a_n\neq 0$, $\sqrt{a_2}>\sqrt{3}(|a_3|+\cdots+|a_n|)^{\frac{1}{2}}$.

(4) $n>4$, $a_4>0$, $a_4^{\frac{1}{4}}>2^{\frac{1}{4}}(1+|a_1|+|a_2|+|a_3|+|a_5|+\cdots+|a_n|)$, $a_n\neq 0$.

(5) The polygon connecting the points $(-1, 0)$, $(0, a_0)$, $(1, a_1)$, $\cdots$, (n, a_n), $(n+1, 0)$ is concave from below at the point $(1, a_1)$ and convex at its other vertices. Condition (1) is an improvement of the theorem due to Perron [Algebra, vol. 2, de Gruyter, Leipzig, 1933, p. 35] that $f(x)$ is irreducible if $|a_1|>1+|a_2|+\cdots+|a_n|$, $a_n\neq 0$, and condition (2) a refinement of the result due also to Perron in which $A=4^{n-1}$. Condition (1) is established by application of Berwald's theorem [Math. Z. **37**, 61–76 (1933)] that under condition (1) $f(x)$ has $n-1$ zeros in $|x|<1$ and condition (2) largely by application of Mayer's theorem [Nouv. Ann. Math. (3) **10**, 111–124 (1891)] that under condition (2) $f(x)$ has $n-2$ zeros in $|x|<1$.

Further theorems are proved on the irreducibility of polynomials of the form

$$f(x)=a_0p^k+a_1x+\cdots+a_nx^n,$$

where a_j are integers, p a prime number and k a positive integer, as well as for polynomials of the form $f(x)=g(x)x+M(x)$, where $g(x)$ and $M(x)$ are polynomials with integral coefficients and with certain prescribed distributions of their zeros. *M. Marden* (Milwaukee, Wis.).

Lipka, Stephan. Über die absolute Konvergenz von Polynomreihen. Math. Z. **49**, 192–196 (1943). [MF 10055]

The author considers the absolute convergence of polynomial series $\sum a_n P_n(z)$ and generalizes some theorems found by the reviewer [Amer. Math. Monthly **45**, 220–226 (1938)]. The $P_n(z)$ have real coefficients. If the zeros of $P_n(z)$ are real and restricted to a fixed interval (ξ_1, ξ_2) and if the series converges absolutely at a point z_0, not on the real axis, then it is also absolutely convergent in the lune $|z-\xi_1|\leqq|z_0-\xi_1|$, $|z-\xi_2|\leqq|z_0-\xi_2|$. It is permitted to let $\xi_1\rightarrow-\infty$ or $\xi_2\rightarrow+\infty$. The author can also allow complex zeros if they are properly limited. In particular, let $N(P_n, a)$ denote the number of zeros of $P_n(z)$ in the half-plane $\Re(z)\leqq a$ and suppose that $N(P, a)=\sup N(P_n, a)<\infty$ for every finite a. If the series converges absolutely at a point $z_0=x_0+iy_0$, $y_0>0$, and every $P_n(z)\neq 0$ in a fixed neighborhood of z_0, then the series is absolutely convergent in the half-plane $\Re(z)>x_0$. *E. Hille* (New Haven, Conn.).

Albert, G. E. and Miller, L. H. Equiconvergence theorems for orthonormal polynomials. Bull. Amer. Math. Soc. **50**, 358–367 (1944). [MF 10599]

Let $\rho(x)$ be nonnegative and integrable on (a, b) (finite interval) and positive on a set of positive measure, so that an orthonormal set of polynomials (ONP) $\{p_n(\rho, x)\}$, $n=0, 1, \cdots$, is determined. A function $f(x)$ has the formal development $f(x)\sim\sum_0^\infty a_n p_n(\rho, x)$. Let $s_n(f, \rho, x)$ denote the nth partial sum. Given two ONP sets corresponding to weight functions $\rho_1(x)$, $\rho_2(x)$ suitably restricted, conditions for equiconvergence:

$$\lim_{n\to\infty}\{s_n(f, \rho_1, x)-s_n(f, \rho_2, x)\}=0$$

are obtained. A typical result is the following. Let $\rho(x)$, $\sigma(x)$ be nonnegative on (a, b), σ being bounded and measurable there, and let a polynomial $\pi(x)$ exist such that $\pi(x)/\sigma(x)$ satisfies a Lipschitz condition

$$(1)\qquad \left|\frac{\pi(x)}{\sigma(x)}-\frac{\pi(t)}{\sigma(t)}\right|<\lambda|x-t|$$

for all x and t on (a, b). Then the equiconvergence condition

$$(2)\qquad \lim_{n\to\infty}\{s_n(f, \rho, \xi)-s_n(f, \rho\sigma, \xi)\}=0$$

holds at every point ξ on (a, b) for which $\pi(\xi)\neq 0$, $|p_n(\rho, \xi)|\leqq H(\xi)$, H being independent of n, provided that $\int_a^b\rho(x)f^2(x)dx$ exists. Moreover, (2) holds uniformly on every closed subset of (a, b) on which $\pi(x)\neq 0$ and for which the bound H is independent of x on the set.
I. M. Sheffer (State College, Pa.).

Doss, S. H. A theorem on uniqueness. J. London Math. Soc. **18**, 137–140 (1943). [MF 10363]

Concerning the definition of a basic series of polynomials, we refer to J. M. Whittaker's "Interpolatory Function Theory" [Cambridge, 1935, chap. 1]. A basic series is called effective in $|z|<R$ if it is capable of representing any function regular in this circle. The author's main result is the proof of the following conjecture of Whittaker. If the basic series is effective in $|z|<R$, the series cannot represent zero in $|z|<R$ unless all the coefficients occurring in the series are zero. *G. Szegö* (Stanford University, Calif.).

Loomis, Lynn H. A short proof of the completeness of the Laguerre functions. Bull. Amer. Math. Soc. **50**, 386–387 (1944). [MF 10603]

Let $L_n(x)$ be the normalized Laguerre polynomial of the nth degree for the weight function e^{-x} on the interval $(0, \infty)$, and let $\varphi_n(x)=e^{-x/2}L_n(x)$. The closure of the system of functions $\varphi_n(x)$ in $L^2(0, \infty)$ is inferred from its closure for the set of characteristic functions $g_b(x)$ of intervals $(0, b)$, the latter property being demonstrated by means of the definition of the functions $\varphi_n(x)$ as coefficients in the power series expansion of the appropriate generating function.
D. Jackson (Minneapolis, Minn.).

Erdös, P. On the maximum of the fundamental functions of the ultraspherical polynomials. Ann. of Math. (2) **45**, 335–339 (1944). [MF 10269]

The author uses the definition

$$(1-2xz+z^2)^{\frac{1}{2}-\alpha}=\sum_{n=0}^{\infty}P_n^{(\alpha)}(x)z^n$$

of the ultraspherical polynomials $P_n^{(\alpha)}(x)$ and proves the following theorem. Let $0\leqq\alpha\leqq\frac{3}{2}$ (in case $\alpha=\frac{1}{2}$ the function $(1-2xz+z^2)^{\frac{1}{2}-\alpha}$ has to be replaced by $(1-2xz+z^2)$). We denote by $x_1, x_2, \cdots, x_n$ the zeros of $f(x)=P_n^{(\alpha)}(x)$ and by M_{kn} the maximum of the absolute value of the fundamental polynomial $f(x)/(f'(x_k)(x-x_k))$ of the Lagrange interpolation in $-1\leqq x\leqq 1$. Then $\max_{1\leqq k\leqq n}M_{kn}=M_{1n}=M_{nn}$ and the latter maxima are attained for $x=\pm 1$. The proof, based on the differential equation of these polynomials, is rather intricate. The theorem, which is in a certain sense the best possible, contains earlier results of Erdös-Grünwald $(\alpha=\frac{1}{2})$ and Webster $(\alpha=\frac{3}{2})$. [Correction. The case $\alpha=\frac{1}{2}$ mentioned on page 335 gives the Tchebycheff and not the Legendre polynomial.] *G. Szegö.*

Subba Rao, M. V. On generalized Legendre polynomials. J. Indian Math. Soc. (N.S.) **7**, 96–101 (1943). [MF 10106]

The generalizations considered by the author are of the form

$$R_n[x; \alpha; Q(x)]=C_nD^{\alpha n}[Q(x)]^n,$$
$$1/C_n=(\alpha n)![(\alpha+1)(\alpha+\beta-1)/2\alpha]^n,$$

which contains several earlier generalizations as special cases. Here α and β are integers, $1\leqq\alpha<\beta$; $Q(x)$ is a polynomial of degree β. The author finds a fairly complicated generating function and also derives recurrence formulas. The case $Q(x)=x^\beta-1$, already considered by P. Kesava Menon [J. Indian Math. Soc. (N.S.) **5**, 92–102 (1941); these Rev. **3**, 116], gives rise to fairly simple formulas.
E. Hille (New Haven, Conn.).

Differential Equations

Glaser, Walter und Lammel, Ernst. Über die Differentialgleichungen zweiter Ordnung, welche lauter Lösungen besitzen, zwischen deren Nullstellen eine projektive Beziehung besteht. Monatsh. Math. Phys. **50**, 289–297 (1943). [MF 10482]

The relation between object and image in electronic optics leads to a differential equation of the form

$$(1)\qquad d^2y/dz^2+F(z)y=0$$

having the following properties: any two subsequent zeros z_0 and z_1 of a solution y of (1) satisfy the projective relation

$$(2)\qquad z_1=\frac{az_0+b}{cz_0+d}, \qquad ad-bc\neq 0;\ c\neq 0.$$

(No fixed point of (2) is supposed to lie between z_0 and z_1.) The authors solve the problem of finding all differential equations with these properties assuming that there exists at least one solution with two subsequent zeros and that all solutions have continuous second derivatives (except for the fixed points of (2)). By a suitable choice of the origin the authors write (2) in the form

$$(3)\qquad z_1=\frac{az_0+b}{z_0+a}$$

and treat separately the cases $b<0$ (elliptic case), $b=0$ (parabolic case) and $b>0$ (hyperbolic case). We quote the result for the elliptic case: if $b=-l^2$ $(l>0)$ the following condition is necessary and sufficient in order that between two subsequent zeros z_0 and z_1 of any solution of (1) the relation (3) holds: $F(z)$ is of the form

$$F(z)=\frac{-l^2}{(z^2+l^2)^2}\left(1+\frac{g''(\xi)}{g(\xi)}\right), \qquad z=l \operatorname{tg} \xi,$$

where $g(\xi)$ is arbitrary except for the following properties: (i) g has continuous second derivatives; (ii) $g(\xi+\alpha)=-g(\xi)$ $(\operatorname{tg}\alpha=-1/a)$; (iii) in each interval $\bar{\xi}\leqq\xi<\bar{\xi}+|\alpha|$, g has one and only one simple zero. For such g the functions

$$(l^2+z^2)^{\frac{1}{2}}g(\xi), \quad (l^2+z^2)^{\frac{1}{2}}g(\xi)\int d\xi/g^2(\xi)$$

form a fundamental system of (1). Corresponding results are derived in the parabolic and hyperbolic case.

E. H. Rothe (Ann Arbor, Mich.).

Whyburn, William M. Differential systems with general boundary conditions. Univ. California Publ. Math. (N.S.) 2 [No. 1, Seminar Rep. in Math. (Los Angeles)], 45–61 (1944). [MF 10454]

The author studies linear differential systems of order n, with n linear boundary conditions related to a set $g=\{d_i\}$ (on (a, b)) of the first species. Use is made of matrix notation, capital letters denoting square matrices of n rows. By a solution on (a, b) is understood a matrix of absolutely continuous elements satisfying the system involved almost everywhere. Integrability is in the Lebesgue sense. It is shown that the problem

$$(1)\qquad Y'+P(x)Y=Q,$$

$$(2)\qquad \sum_1^\infty A_iY(d_i)+\int_a^b F_1(x)Y(x)dx=C_1$$

(F_1 integrable; A_i, C_1 matrices of constants; $A_1+A_2+\cdots$ absolutely convergent) is equivalent to the problem (1),

$$(3)\qquad AY(a)+BY(b)+\int_a^b F(x)Y(x)dx=C$$

(F integrable; A, B, C matrices of constants). Also, adjoint relationships and Green's function are studied; the derivation of the latter involves a novel method.

W. J. Trjitzinsky (Urbana, Ill.).

Wilkins, J. Ernest, Jr. On the growth of solutions of linear differential equations. Bull. Amer. Math. Soc. 50, 388–394 (1944). [MF 10604]

The paper deals with the differential equation

$$y^{(n)}+\sum_{i=1}^n A_i(x)y^{(n-i)}=B(x),$$

with coefficients $B(x)$ and $A_i(x)$, $i=1, 2, \cdots, n$, that are continuous on $0\leqq x<\infty$ and such that $\int_0^\infty B(x)dx$ exists. It is shown that $\lim_{x\to\infty} y^{(n-1)}(x)$ exists for any solution $y(x)$, under either of the following sets of conditions: (i) that

$$\int_0^\infty x^{i-1}|A_i(x)|dx, \qquad i=1, 2, \cdots, n,$$

exist; (ii) that

$$\int_0^\infty x^{i-1}\{|A_i|-A_i\}dx, \quad i=1, 2, \cdots, n,$$

exist, and

$$\limsup_{x\to\infty} x^{i-k-1}\int_x^\infty t^k\{|A_i|+A_i\}dt<\frac{4(i-k-1)!k!}{n(n-1)},$$

for $i=2$, $k=0$, or $i=2j-1, 2j$; $k=i-j, \cdots, i-2$; $j=2, \cdots, (n+1)/2$. If, in addition,

$$\int_0^\infty \sum_{i=1}^n x^{i-1}\{|A_i(x)|+A_i\}dx=\infty,$$

then $\lim_{x\to\infty} y^{(n-1)}(x)=0$.

R. E. Langer (Madison, Wis.).

Hölder, Ernst. Einordnung besonderer Eigenwertprobleme in die Eigenwerttheorie kanonischer Differentialgleichungssysteme. Math. Ann. 119, 21–66 (1943). [MF 10094]

The first part of this paper is concerned with the existence of normal solutions for a boundary value problem

$$dx_i/dt=\sum_{j=1}^n [A_{ij}(t)+\lambda B_{ij}(t)]x_j,$$

$$\sum_{j=1}^n [M_{ij}x_j(t^0)+N_{ij}x_j(t^1)]=0, \quad i, j=1, \cdots, n,$$

under the following hypotheses: (1) the coefficients $A_{ij}(t)$, $B_{ij}(t)$ are continuous on $t^0\leqq t\leqq t^1$, and the $n\times 2n$ matrix $\|M_{ij}, N_{ij}\|$ is of rank n; (2) the system is self-adjoint under the transformation $z_i=\sum_j T_{ij}(t)x_j$; (3) the matrix $S(t)=\|\sum_k T_{ki}(t)B_{kj}(t)\|$ is symmetric and positive semi-definite on $t^0\leqq t\leqq t^1$; moreover, $S(t)$ is of constant rank $n-p$, $0\leqq p<n$, on this interval. If a further condition of normality is satisfied by such a system it is definitely self-adjoint in the sense defined by Bliss [Trans. Amer. Math. Soc. 44, 413–428 (1938)]; in the general case considered by the author, however, the system may possess abnormal solutions. The consideration of the above system is reduced to that of a canonical system involving $2n$ linear differential equations and boundary conditions, and the determination of normal solutions of this canonical system is shown to be equivalent to the solution of a pair of adjoint vector integral equations. The author obtains expansion theorems and establishes certain minimizing properties of the characteristic values. The second part of this paper is devoted to a boundary value problem associated with the second variation of a problem of Bolza in the calculus of variations. The system considered involves the characteristic parameter

in a manner analogous to that utilized by Lichtenstein in the treatment of boundary value problems associated with the second variation of the simpler problems of the calculus of variations. In the third part of the paper the author treats certain boundary value problems involving a self-adjoint linear differential equation of even order. The author was evidently unaware of a recent paper by the reviewer [Trans. Amer. Math. Soc. **52**, 381–425 (1942); these Rev. **4**, 100], which contains results having many points of contact with those herein obtained. *W. T. Reid* (Chicago, Ill.).

Sussholz, B. Forced and free motion of a mass on an air spring. J. Appl. Mech. **11**, A-101–A-107 (1944). [MF 10615]

A nonlinear system consisting of a light piston confining a column of air in a tube is studied by considering the differential equation $m\ddot{x}+E(x)=F(t)$. Here $E(x)$ is nonlinear due to the nonlinear elasticity of the air column. The author chooses a physically appropriate $E(x)$ and then considers the case where $f(t)=0$, the case where $F(t)$ is a step function (pressure wave) and the case where $F(t)$ is a rectangular pulse. The last case is used to give approximate results for any $F(t)$ simply by approximating to such an $F(t)$ by a rectangular step function. Application is made to ballistics and bomb explosions. *N. Levinson* (Cambridge, Mass.).

Jouguet, Émile. Application de la méthode de la variation des constantes à l'étude de oscillations non linéaires. C. R. Acad. Sci. Paris **217**, 218–220 (1943). [MF 10644]

The author observes that certain assumptions frequently made in obtaining the first perturbation term in solving a nonlinear differential equation of the second order lead to errors in this term of the order of magnitude of the term itself and shows how this may be avoided. *N. Levinson.*

Feshbach, Herman. On the perturbation of boundary conditions. Phys. Rev. (2) **65**, 307–318 (1944). [MF 10692]

The partial differential equation $\nabla^2\phi+k^2\phi=0$ is considered in connection with a boundary condition of type $\partial\phi/\partial n=F\phi$, where F is a function which may vary on the surface and n denotes the normal drawn outwards. Special attention is given to the cases in which F is either small or large. With the aid of the Green's function for either the condition $\partial\phi/\partial n=0$ or $\phi=0$ the problem for a simple surface can be reduced to that of the solution of a homogeneous integral equation in which the value of ϕ at the point of observation is expressed in terms of the surface values of ϕ. Methods of reducing the equation to an ordinary integral equation are considered in which allowance is made for the fact that ϕ or its slope is discontinuous at the surface. Approximate values of ϕ and of the eigenvalues k^2 are obtained with the aid of an orthonormal set of functions for the partial differential equation, a region R_0 including the region R under consideration and a convenient boundary condition on the boundary S_0 of R_0.

A vector wave equation $\nabla\nabla\cdot v+k^2v=0$ and related boundary conditions $\nabla\cdot v=-k^2/F(n\cdot v)$ require the use of a vector form of Green's theorem and a Green's dyadic. First and higher approximations for both eigenvalues and eigenfunctions are given. An associated secular equation is found to be Hermitian. Expansions for the Green's dyadic are used. Integral equations for the Green's function and dyadic are given and a method of solution in series of eigenfunctions is indicated. *H. Bateman* (Pasadena, Calif.).

Magnaradze, L. G. The Dirichlet problem as a limiting case of the Cauchy-Dirichlet problem for the wave equation, heat conduction, and similar equations. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **11**, 73–96 (1942). (Russian. Georgian summary) [MF 10283]

Let $U(P, t)$ be a solution of the differential equation

$$\nabla^2U=\alpha\frac{\partial^2U}{\partial t^2}+2\beta\frac{\partial U}{\partial t}+X(x, y, z, t)$$

in a region D with boundary S, satisfying the initial conditions $U(P, t_0)=\varphi_1(P)$, $U_t(P, t_0)=\varphi_2(P)$, for P in $D-S$, and the boundary conditions $\lim_{P\to M} U(P, t)=f(M, t)$ for M on S. The author shows that, with certain restrictions imposed on α, β, φ_1, φ_2, X, and f, the limit $\lim_{t\to\infty}U(P, t)=u(P)$ exists and that $u(P)$ satisfies Laplace's equation in $D-S$ and the boundary condition $\lim_{P\to M}U(P)=f(M)=\lim_{t\to\infty}f(M, t)$ for M on S.

For $\alpha>0$ his assumptions are: $\beta>0$; φ_1 and φ_2 have continuous partial derivative to the fourth order in D; $f(M, t)$ is continuous on S, has continuous derivatives to the fifth order with respect to t and approaches the limit $f(M)$ as t becomes infinite; X has continuous derivatives to the third order with respect to x, y, z, t. In addition he imposes requirements making the initial conditions consistent with the differential equation and with the boundary conditions. For $\alpha=0$, $X=0$, the problem is simpler. The initial condition involving φ_2 drops out and $f(M, t)$ need have continuous derivatives in t to the third order only. *W. E. Milne.*

Hornich, Hans. Zwei vermischte Randwertaufgaben der Potentialtheorie. Monatsh. Math. Phys. **50**, 40–47 (1941). [MF 10491]

In a previous paper [Monatsh. Math. Phys. **41**, 7–19 (1934)] the author has solved the mixed boundary-value problem of determining the potential function u in the upper half-plane for which $\partial u/\partial y+\lambda\partial u/\partial x$ takes on given values on a finite number of intervals of the real axis, λ being given and constant in each interval, and such that u takes on prescribed values on the rest of the axis. The solution was obtained through a Green's function. In the present paper the author considers the same problem with λ taken as any function of class C^1. The generalization of the earlier Green's function yields first the solution for the case where u and $\partial u/\partial y+\partial(u\lambda(x))/\partial x$ are alternately given. Then this solution, together with a new Green's function, yields the desired solution. The function u is unique and is given by an integral formula involving only known functions. *L. H. Loomis* (Cambridge, Mass.).

Muschelišvili, N. I. On the fundamental mixed boundary value problem in the theory of the logarithmic potential for multiply connected domains. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] **2**, 309–313 (1941). (Russian) [MF 10289]

The author designates by S a connected region in the complex plane, bounded by simple closed nonintersecting contours L_0', $\cdots$, $L_{m'}'$, L_1'', $\cdots$, $L_{m''}''$, of which L_0' contains the rest; $L'=L_0'+\cdots+L_{m'}'$, $L''=L_1''+\cdots+L_{m''}''$, $L=L'+L''$. The following fundamental boundary problem of the logarithmic potential is solved. To find a single valued function $\varphi(z)=u+iv$, analytic in S, so that u, v are continuous in $S+L'$ and $S+L''$, respectively, subject to the conditions $u=f+a_j$ (on L_j'), $v=g+b_j$ (on L_j''), where f, g are preassigned continuous real functions, while real constants a_j, b_j are to be determined. If one puts, say, $a_0=0$,

the solution is determinate and can be expressed in the form

$$\varphi(z)=\frac{1}{\pi i}\int_{L'}\frac{\mu dt}{t-z}+\frac{1}{\pi}\int_{L''}\frac{\nu dt}{t-z},$$

where μ, ν satisfy a system of Fredholm integral equations of the second kind. *W. J. Trjitzinsky* (Urbana, Ill.).

Landkoff, N. On the position of irregular points in the generalized problem of Dirichlet. C. R. (Doklady) Acad. Sci. URSS (N.S.) **39**, 335–337 (1943). [MF 10754]

Let Ω be a three dimensional region with boundary F such that every Jordan curve on F can be reached from the outside by a surface. Let A be a Jordan curve on F with the property that every point of A with the exception possibly of a set of the first category is attainable by a cone. Also let the projection of A on a plane give a curve with only a denumerable infinitude of multiple points. The object of this note is to prove that the irregular points on A form a set of first category.

The proof is based on the criterion of Wiener, Kellogg, and Vasilesco that P be a regular point with respect to the Dirichlet problem: that the integral $\int_0 C(\rho, P)d\rho/\rho^2$ diverge, where $C(\rho, P)$ is the capacity of the intersection of a sphere of radius ρ and center P with the complement of Ω. Using this result, the author is able to reduce the problem to the following one: given a Jordan surface S with edge A, prove that the regular points for the solution of the outer problem of Dirichlet lie everywhere dense on A.

The theorem is first proved for the case when A is a plane curve and S the plane area bounded by A, the proof in this case being very simple. A geometrical argument and the fact that the capacity of a set does not increase when the set is projected on a plane allow the extension to the case when A is plane but S is any Jordan surface having A as edge. The proof of the general case is carried out by a process based on projecting A on a suitable plane. The proof is somewhat sketchy and in one place at least a portion of the manuscript seems to have been omitted in printing. Consequently the arguments are unclear in places and not thoroughly understood by the reviewer. *J. W. Green.*

Jaeger, J. C. Note on a problem in radial flow. Proc. Phys. Soc. **56**, 197–203 (1944). [MF 10740]

The author shows that, if one calculates the temperature distribution at a time t in a cylinder of radius a with (1) the initial temperature and (2) the normal derivative of the temperature on the boundary of the cylinder assigned constants, this distribution can be approximated with sufficient accuracy for most physical purposes by the distribution in a slab of width a, with the same initial and boundary conditions at the corresponding time t_1. Thus for Ht/a^2 (H a physical constant) sufficiently large, the simple approximate relation

$$\frac{Ht_1}{a^2}=\frac{2Ht}{a^2}-\frac{1}{12}$$

exists. More complicated approximate relations exist for Ht/a^2 small. *A. E. Heins* (Cambridge, Mass.).

***Churchill, Ruel V. Modern Operational Mathematics in Engineering.** McGraw-Hill Book Co., Inc., New York, 1944. x+306 pp. $3.50.

This textbook deals with the applications of the Laplace transformation to the solution of ordinary and partial linear differential equations. These differential equations find their roots in elementary theoretical physics and the analytical phases of the engineering sciences.

The book is divided into two parts. The first part offers an elementary treatment of the Laplace transformation; that is, no explicit use is made of the theory of functions of a complex variable and as such a more complete development of the Laplace transformation is relegated to later chapters. In these first four chapters which make up the elementary treatment we find two chapters devoted to elementary properties of the Laplace transformation and one devoted to the solution of certain ordinary linear differential equations which arise in vibration theory and similar problems. The fourth chapter discusses the application of the Laplace transformation to the solution of a number of elementary boundary value problems in partial differential equations such as one might encounter in the dynamics of continuous media and the theory of heat conduction.

The second portion embarks upon a more advanced program. Chapters V and VI discuss the elements of functions of a complex variable and the inversion formula of the Laplace transformation. With this additional background, the author is ready to go forward and solve linear partial differential equations with constant coefficients which give rise to transforms whose inverses cannot be handled in an elementary fashion. Thus, in chapters VII and VIII a number of problems in the theory of heat conduction and vibrations are discussed. Here many of the problems have a solution in series form, and the author takes pains to demonstrate how one may show that the derived solution does indeed satisfy the differential equation, the initial and boundary conditions. The book closes with a chapter on Sturm-Liouville systems and one on finite Fourier transforms. A table of Laplace transforms is given in the appendix. *A. E. Heins* (Cambridge, Mass.).

Integral Equations

Iglisch, Rudolf. Bemerkungen zu einigen von Herrn Collatz angegebenen Eigenwertabschätzungen bei linearen Integralgleichungen. Math. Ann. **118**, 263–275 (1941). [MF 10706]

The author makes some remarks which improve the earlier analysis of Collatz [Math. Z. **46**, 692–708 (1940); these Rev. **2**, 312]. *A. E. Heins* (Cambridge, Mass.).

Wilkins, J. Ernest, Jr. Definitely self-conjugate adjoint integral equations. Duke Math. J. **11**, 155–166 (1944). [MF 10156]

The author considers a system of integral equations

$$y_i(x)=\lambda\int_a^b\sum_j K_{ij}(x,t)y_j(t)dt,\quad i,j=1,\cdots,n,$$

under the following hypotheses: (1) the matrix kernel $K(x,t)\equiv\|K_{ij}(x,t)\|$ is of the form $K(x,t)=H(x,t)S(t)$, where $S(x)$ is an Hermitian matrix with continuous elements, and the elements of the matrix $H(x,t)$ are bounded and have their discontinuities regularly distributed on $a\leqq x\leqq b$, $a\leqq t\leqq b$; (2) $S(x)K(x,t)=K^*(t,x)S(t)$, where K^* denotes the conjugate transpose of K; (3) the Hermitian

form $v^*S(x)v$ is nonnegative on $a \leqq x \leqq b$. For such systems it is shown that the characteristic values are real, and the index of a characteristic value is equal to its multiplicity as a zero of the Fredholm determinant; extremizing properties of the characteristic values are proved, and expansion theorems are established. These results are decided extensions of those on definitely self-adjoint integral systems established by Reid [Trans. Amer. Math. Soc. 33, 475–485 (1931)]. In conclusion, the author considers a special definitely self-conjugate adjoint integral system analogous to a special differential system treated by Reid [Trans. Amer. Math. Soc. 52, 381–425 (1942), in particular, §8; these Rev. 4, 100]. *W. T. Reid* (Chicago, Ill.).

Chvoles, A. R. On Fredholm's integral equations of the third kind. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 2, 389–395 (1941). (Russian. Georgian summary) [MF 10292]

The author gives a solution of Picard's equation

$$(1) \qquad \varphi(x)+\lambda\int_a^b (K(x,s)/A(s))\varphi(s)ds=f(x),$$

distinct from that of Picard and under conditions on K, f, A lighter than those involved in Picard's paper [Ann. École Norm. Sup. (3) 28, 313–324 (1911)]. Without any essential loss of generality, the equation is taken in the form

$$(2) \qquad \varphi(x)+\lambda\int_a^b (K(x,s)/s)\varphi(s)ds=f(x), \qquad a<0<b.$$

Convergence of Picard's limiting process and existence of a solution is established under the supposition that $K(x,s)$, $f(x)$ have first order derivatives with respect to x and s, satisfying the Lipschitz condition. *W. J. Trjitzinsky.*

Vekua, Ilja. On a class of singular integral equations with integrals in the sense of Cauchy's principal value. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 2, 579–586 (1941). (Russian. Georgian summary) [MF 10296]

The equation studied is

$$(A) \qquad A\varphi\equiv\alpha(x)\varphi(x)-\int_L (k(x,t)/(t-x))\varphi(t)dt=f(x),$$

where L is the frontier of a multiply connected region bounded by nonintersecting simple contours $L_0, \cdots, L_m$ (L_0 containing the rest), x, t are on L, the α, k, f are given of class H (Hölder) on L and the unknown φ is sought in H. Moreover, (A′) $A'\psi=f(x)$ is the equation associated with (A). It is assumed that $\beta(x)=k(x,x)$ is in H; $c^2=\alpha^2+\pi^2\beta^2\neq0$; $2\pi n_j=\arg[(\alpha+i\pi\beta)/(\alpha-i\pi\beta)]_{L_j}$; $n=n_0+\cdots+n_m=$ index of (A). The following functions are formed:

$$\gamma=(\alpha+i\pi\beta)/(\alpha-i\pi\beta),$$

$$2\pi i\omega=\int_L \log\left(\gamma(t)t^{-n}\prod_1^m(t-a_k)^{n_k}\right)\frac{dt}{t-x},$$

$$c\gamma_n=x^{n/2}e^{-\omega(x)}\prod_1^m(x-a_k)^{n_k/2},$$

a_k interior to L_k, $\beta^*(x)=\beta(x)c^{-2}(x)$. Previously it has been shown that (A) is equivalent to a certain regular Fredholm equation (F), whose left member is $A_n{}^*\varphi$ (if $n\geqq0$); if $n<0$, to (F) there are adjoined certain integral relations. Now $A_n{}^*\varphi=0$ has p solutions $\varphi_1, \cdots, \varphi_p$; the solutions of the associated equations are $\psi_1, \cdots, \psi_p$. Let $A_{jk}=\int_L\beta^*x^{k-1}\gamma_n{}^{-1}\psi_j dx$. The rank of (A) is defined as $r=p-q$, where q is the rank of the matrix (A_{jk}). The following is proved. (I) If $n\geqq0$ and $r=0$, (A) has a solution for every f; if $r>0$, (A) has no solution in general. Moreover, $A\varphi=0$ has $n+r$ solutions. (II) If $n<0$, $A\varphi=f$ has a solution only if f satisfies certain relations of orthogonality; $A\varphi=0$ has a number s of solutions, where s can be determined. (III) $A\varphi=f$ has a solution if and only if f is orthogonal to to the solutions of $A'\varphi=0$. *W. J. Trjitzinsky.*

Vekua, Ilja. On the reduction of singular integral equations to equations of Fredholm's type. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 2, 697–700 (1941). (Russian. Georgian summary) [MF 10300]

In this note the author makes a supplement to the paper reviewed above. It is proved that the equation (F), with the adjoined integral relations (when $n<0$), can be replaced by a single Fredholm equation, the adjoined relations containing f but not φ. *W. J. Trjitzinsky* (Urbana, Ill.).

Vekua, Ilja. On the theory of singular integral equations. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 3, 869–876 (1942). (Russian. Georgian summary) [MF 10320]

The author studies operators

$$A\varphi\equiv\alpha(x)\varphi(x)-i\pi\beta(x)E\varphi(x)-\lambda K\varphi(x),$$

where α, β are H-functions (that is, are of Hölder class), λ is a parameter and

$$\pi iE\varphi=\int_L\varphi(t)dt/(t-x), \quad K\varphi\equiv\int_L K(x,t)\varphi(t)dt,$$

x on L, φ in H; it is assumed that $K(x,t)=K_0(x,t)|t-x|^{-\nu}$, $\nu<1$, where $K_0(x,t)$ is H in x and in t; $\alpha^2+\pi^2\beta^2\neq0$ on L. The index n of A is defined as indicated in a previously reviewed paper [see the second preceding review]. The fundamental problem, according to the author, for (A) $A\varphi=f$ is to find a Fredholm equation $M\varphi=g$ equivalent to (A). The earlier point of view was more narrow: namely, to find an operator P so that $PA\varphi=Pf$ is a Fredholm equation equivalent to (A). Under the old point of view the problem has a solution for all f if and only if $n>0$. With the aid of an ingenious method the author shows that under the new formulation the problem of equivalence is always soluble. *W. J. Trjitzinsky* (Urbana, Ill.).

Kupradze, V. D. On the theory of integral equations with integrals in the sense of Cauchy's principal values. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 2, 587–596 (1941). (Russian. Georgian summary) [MF 10297]

The author studies the equation

$$K\varphi(s)\equiv a(s)\varphi(s)-\lambda\int_\gamma[b(s)/(t-s)+K(s,t)]\varphi(t)dt=f(s),$$

where γ is a smooth closed curve, s, t are on γ, the a, b, K, f are of class H on γ and the unknown φ is sought in H. The purpose of this note is to present a systematic treatment in order to bring out more clearly than heretofore the connection between the various facts of the theory and to simplify some of the known proofs. *W. J. Trjitzinsky.*

Kupradze, V. D. On a problem of equivalence in the theory of singular integral equations. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 2, 793–798 (1941). (Russian. Georgian summary) [MF 10302]

The author continues the developments of the note reviewed above and considers the problem of finding a linear operator R so that (1) $RK\varphi=Rf$ is a Fredholm equation of the second kind all of whose solutions satisfy (2) $K\varphi=f$. This problem is easily soluble if the index $n\geqq 0$. The author effectively constructs R when $n<0$.

W. J. Trjitzinsky (Urbana, Ill.).

Muscheliŝvili, N. I. and Kveselava, D. A. Singular integral equations with Cauchy-type kernels on open contours. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 11, 141–172 (1942). (Russian. Georgian summary) [MF 10286]

The authors study the equation

$$T\varphi=\alpha(t_0)\varphi(t_0)+(1/\pi)\int_L k(t_0,t)\varphi(t)dt/(t-t_0)=f(t_0),$$

where L consists of a number of open, smooth nonintersecting arcs, k, α, f are assigned essentially of class H on L and φ is unknown, in greater detail than in an earlier work (M) [N. I. Muscheliŝvili, same Trav. 10, 1–43 (1941); these Rev. 4, 160]. The solutions are sought in the class H (Hölder class) interior to the arcs; near the end points c of the arcs the solutions are to satisfy conditions essentially as in (M); it is required that at the points c the orders of infinity of a solution be less than unity. All solutions bounded near preassigned points c constitute a class of solutions, defined by these points. Points c for which $\Re((1/2\pi i)\log A(c))$ is not an integer are termed regular (here $A=(\alpha-i\beta)/(\alpha+i\beta)$; $\beta(t)=k(t,t)$). The class of solutions defined by preassigned regular points c is the totality of solutions bounded near these c. The class defined by regular points $c_1,\cdots,c_k$ is designated by $h(c_1,\cdots,c_k)$ or just h. Classes $h=h(c_1,\cdots,c_k)$ and $h'=h(c_{k+1},\cdots,c_m)$ are termed associated. With a_j, b_j denoting initial and terminal c points, respectively, let $2\pi i\alpha_j=-\log A(a_j)$, $2\pi i\beta_j=\log A(b_j)$. Integers μ_j, ν_j are defined so that $-1<\Re\alpha_j+\mu_j$, $\Re\beta_j+\nu_j<1$. The index is the number $\mathrm{H}=-\sum(\mu_j+\nu_j)$. Some of the results are as follows. (I) The number of solutions of $T\varphi=0$ is finite. (II) $T\varphi=f$ is soluble in class h if and only if f is orthogonal to the ψ_j, where the ψ_j form a complete system (of associated class h') satisfying the associated equation $T'\psi=0$. (III) If l is the number of solutions of class h of $T\varphi=0$ (H denoting the index of this class) and if l' is the number of solutions of the associated class h' of $T'\psi=0$, then $l-l'=\mathrm{H}$. *W. J. Trjitzinsky* (Urbana, Ill.).

Muscheliŝvili, N. I. Systems of singular integral equations with kernels of Cauchy's type. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 3, 987–994 (1942). (Russian. Georgian summary) [MF 10321]

Under consideration is the system

$$\text{(1)}\quad \Lambda_\alpha\varphi\equiv\sum_\beta a_{\alpha\beta}(t_0)\varphi_\beta(t) +(1/\pi i)\int_L\sum_\beta k_{\alpha\beta}(t_0,t)\varphi_\beta(t)dt/(t-t_0)=f_\alpha(t_0),$$

where $\alpha=1,\cdots,n$; L is a simple closed smooth plane contour, t_0, t are on L, the $a_{\alpha\beta}$, $k_{\alpha\beta}$, f_α are preassigned on L as functions H (that is, of Hölder class), integrations are in the sense of Cauchy principal values and solutions $\varphi_\alpha(t)$ are sought in H. The author generalizes the main results of F. Noether [given for $n=1$ in Math. Ann. 82, 42–63 (1921)] to the case $n>1$. It is proved that in order that (1) should have a solution it is necessary and sufficient that $\int_L\psi^{(\sigma)}fdt=0$, $\sigma=1,\cdots,l'$, where $\psi^{(\sigma)}=(\psi_1^{(\sigma)},\cdots,\psi_n^{(\sigma)})$ is a complete system of solutions of the homogeneous associated system $\Lambda_\alpha'\psi=0$. Also, if l and l' denote the numbers of linearly independent solutions of $\Lambda_\alpha\varphi=0$ and $\Lambda_\alpha'\psi=0$, respectively, then the difference $l-l'$ depends on the functions $a_{\alpha\beta}(t_0)$, $k_{\alpha\beta}(t_0,t_0)$ only. *W. J. Trjitzinsky* (Urbana, Ill.).

Calculus of Variations

Karlin, Meyer. Characterization of the system of extremals of variation problem of higher order in the plane. I. J. Math. Phys. Mass. Inst. Tech. 22, 158–180 (1943). [MF 9256]

It is a classical result of Darboux that the system of curves defined by the differential equation $y''=F(x,y,y')$ represents the totality of extremals of a calculus of variations problem $\int\varphi(x,y,y')dx=\min$. The present paper considers the case of higher order variational problems in the plane, for example,

$$\text{(1)}\qquad \int\varphi(x,y,y',y'')dx=\min.$$

The Euler equation is a fourth order equation, which when solved for $y^{(\mathrm{iv})}$ has the form

$$\text{(2)}\qquad y^{(\mathrm{iv})}=A+By'''+Cy'''^2,$$

where A, B, C are functions of x,y,y',y''. The present paper considers the inverse problem of determining the conditions under which a differential equation of type (2) arises from a calculus of variations problem of type (1). The author obtains a set of six equations for the functions A, B, C which forms the necessary and sufficient conditions that (2) arise from (1). The function φ is then determined uniquely, except for an additive exact derivative $(d/dx)\psi(x,y,y')$. One of the methods used is to set down the Euler equation for (1), identify it with (2) resulting in a set of partial differential equations for φ and write down the conditions that this set of equations be solvable for φ. Numerous examples are given. A geometric property of the family of extremals for higher order problems is stated, depending on the special form of the Euler equation. *M. Shiffman.*

Ewing, George M. Minimizing an integral on a class of continuous curves. Duke Math. J. 10, 471–477 (1943). [MF 8968]

Let K_1 be the class of all continuous curves on a compact subset A of an Euclidean space and let K_2 be the class of all absolutely continuous curves in K_1. Let $J(C)$ be the usual calculus of variations integral on K_2. For each curve C in K_1 let $J^*(C)=\text{g.l.b}\ J(C)$ for all sequences $\{C_n\}$ in K_2 converging to C in the Fréchet sense. The author studies conditions under which a minimizing curve for $J^*(C)$ exists on a subclass K of K_1. He also studies conditions under which $J(C)$ is lower semicontinuous on a rectifiable curve C_0.

M. R. Hestenes (Chicago, Ill.).

Ewing, George and Morse, Marston. The variational theory in the large including the non-regular case. I. Ann. of Math. (2) 44, 339–353 (1943). [MF 8866]

The present paper is the first of two papers dealing with the variational theory in the large for nonregular single integral problems in the calculus of variations [cf. the following review]. Its principal purpose is to present known relevant material in a form that will lead to the general theorems given in the second paper. In particular, the authors give an extensive discussion of the consequences of the assumptions of convexity and positive seminormality on the integrand function $f(x, r)$. New proofs are given which are in general independent of differentiability properties of f. These results are used in order to determine conditions on the integrand f which are sufficient to insure the bounded compactness of the integral $J=\int f(x, \dot{x})dt$ and the equivalence of convergence in length and J-length. Particular attention is paid to the determination of the minimum hypotheses under which the results are valid. The concept of a pseudo-limiting curve is introduced. This concept is useful in the simplification of the theorems and proofs given in the second paper. *M. R. Hestenes.*

Ewing, George and Morse, Marston. The variational theory in the large including the non-regular case. II. Ann. of Math. (2) 44, 354–374 (1943). [MF 8867]

[Cf. the preceding review.] Let $\alpha, \beta, \cdots$ be arcs on a metric space M and let $J(\alpha)$, $L(\alpha)$ be functions on M. In application to the calculus of variations α is an arc, $L(\alpha)$ is the length of α and $J(\alpha)$ is the usual integral in the calculus of variations. Besides the metric $\alpha\beta$ of M, the authors introduce two metrics $|\alpha\beta|=\alpha\beta+|L(\alpha)-L(\beta)|$ and $(\alpha\beta)=\alpha\beta+|J(\alpha)-J(\beta)|$. These define metric spaces L and J, respectively. The concept of J-deformations and upper reducibility are defined in terms of these metrics. It is emphasized that the concept of upper reducibility is weaker than that of upper semicontinuity and that it plays an important role in the calculus of variations in the large. It is shown that, under the usual hypotheses used for existence theorems for a minimizing arc, the integral $J(\alpha)$ is upper-reducible on L at each rectifiable curve β of positive length.

Defining an extremal to be an arc β for which the first variation of J is zero for all admissible variations whose values vanish, it is shown that an arc β is an extremal if it is homotopically critical and if convergence in J-length to β implies convergence in L-length to β. These results and others are first obtained for arcs embedded in a Euclidean region and then extended, by means of an auxiliary lemma, to arcs on a compact Riemannian manifold.
M. R. Hestenes (Chicago, Ill.).

Wilkins, J. Ernest, Jr. Multiple integral problems in parametric form in the calculus of variations. Ann. of Math. (2) 45, 312–334 (1944). [MF 10268]

This paper is concerned with the calculus of variations problem of minimizing a multiple integral

$$\int\cdots\int f(y^1, \cdots, y^n, p_1^1, \cdots, p_m^n)dt_1\cdots dt_m,$$

where $p_a^i=\partial y^i/\partial t_a$ ($i=1, \cdots, n$; $a=1, \cdots, m<n$), in a class of varieties with common boundary and represented parametrically by equations of the form $y^i=y^i(t)$. The author first derives consequences of the homogeneity condition that is necessary and sufficient for the integral to be independent of the parametric representation; for the special case $n=m+1$ certain additional consequences of this homogeneity condition are deduced. Necessary conditions for a minimum are discussed; in particular, there is presented a Weierstrass condition in terms of a new E-function which has advantages over the ordinary E-function and the one that has been proposed by Carathéodory. A field for the considered problem is defined, and necessary and sufficient conditions for the invariance of an integral are established. *W. T. Reid* (Chicago, Ill.).

Lichnerowicz, André. Sur une extension du calcul des variations. C. R. Acad. Sci. Paris 216, 25–28 (1943). [MF 9998]

The author defines, for a certain class of functionals F attached to a curve defined parametrically in curved n-space, the notion of subdifferential of F and also the related notion of the subvariation of an integral involving F. This leads to a generalized Euler equation and is applied to the results of two previous notes [C. R. Acad. Sci. Paris 212, 328–331 (1941); 214, 599–601 (1942); these Rev. 3, 63; 4, 226] dealing with Finsler spaces and generalized relativity. An error in the second of these notes is corrected.
O. Frink (State College, Pa.).

Functional Analysis

Pinsker, A. On normed K-spaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) 33, 12–14 (1941). [MF 9618]

A K-space (a linear semi-ordered space of Kantorovitch) is said to be continuous if any positive element y can be represented as a sum of two disjoint positive elements. The space is said to be discrete if for every positive z there is a z' with $0<z'\leqq z$ which cannot be represented as the sum of two disjoint positive elements. A K-space is said to be of type B_2 in case (1) $|x_1|<|x_2|$ implies $\|x_1\|<\|x_2\|$, (2a) if $x_n\downarrow 0$ then $\|x_n\|\to 0$, (2b) if $x_n\uparrow\infty$ then $\|x_n\|\to\infty$. It is stated that every continuous separable K-space of type B_2 is isomorphic with a normal subspace of L, that every discrete separable K-space of type B_2 is isomorphic to a normal subspace of l, and that every K-space of type B_2 is isomorphic to a normal subspace of the product (L, l). Any K-space of type B_2 has the property that, for every sequence (E_n) of sets E_n of positive elements for which $\lim_{n\to\infty}(\sup E_n)=x$, there exist finite sets $E_n'\subset E_n$ such that $\lim_{n\to\infty}(\sup E_n')=x$. A number of other results are stated without proof. *N. Dunford* (New Haven, Conn.).

Ficken, Frederick A. Note on the existence of scalar products in normed linear spaces. Ann. of Math. (2) 45, 362–366 (1944). [MF 10273]

The author proves that, for a normed linear space to permit the definition of a scalar product, it is necessary and sufficient that $|\lambda x+\mu y|=|\mu x+\lambda y|$ whenever $|x|=|y|$. The proof involves the Jordan-von Neumann parallelogram condition $|x+y|^2+|x-y|^2=2|x|^2+2|y|^2$; reference is also made to the reviewer's condition that "perpendicularity" be symmetric. *G. Birkhoff* (Cambridge, Mass.).

Taylor, Angus E. Conjugations of complex linear spaces. Univ. California Publ. Math. (N.S.) 2 [No. 1, Seminar Rep. in Math. (Los Angeles)], 85–102 (1944). [MF 10457]

The author investigates the manner in which complex linear spaces are decomposable into ordered pairs of real

linear spaces, as in the case of the ordinary complex number space. The discussion in the first part of the paper is algebraic and is concerned primarily with operations T on a complex linear space E to E such that $T(z+w)=Tz+Tw$, $Taz=\bar{a}Tz$, $T^2z=z$; these are called conjugations of E and determine the desired decompositions. If M is the set of points for which $Tz=z$, then M is a real linear manifold such that every z in E is uniquely expressible as $x+iy$ with x and y in M. A set M is an essentially real linear manifold if it is real and if x and ix in M implies $x=0$; it is maximal if it is not a proper subset of such a manifold. It is proved that every maximal essentially real linear manifold M gives rise to a unique decomposition of every z in E in the form $x+iy$, with x, y both in M, and that $Tz=x-iy$ is a conjugation of E. The case in which M is not maximal is also investigated. The author shows that for every z_0 in E there is a conjugation of E such that $Tz_0=z_0$.

In the second part of the paper E is assumed to be a complex Banach space and a conjugation of E must now be continuous. A necessary and sufficient condition that an operation on E to E having the algebraic properties of a conjugation be continuous is that the manifold M for which $Tx=x$ be closed and real. If T is a conjugation of E and if E^* is the conjugate space to E, then $T'f=\overline{(f(Tz))}|z \text{ in } E)$ is a conjugation of E^* and $\|T\|=\|T'\|$. If M is the "real part" of E relative to T and M' the "real part" of E^* relative to T', then, if x is in M and f is in M', $f(x)$ is real. This property characterizes M and M'. Necessary and sufficient conditions are given for the existence of a conjugation of E. Finally, the author considers spaces with denumerable bases, complex spaces $E\times E$ and Hilbert spaces and obtains results of considerable interest. *H. H. Goldstine.*

Dieudonné, Jean. Sur le nombre de dimensions d'un module. C. R. Acad. Sci. Paris **215**, 563–565 (1942). [MF 10188]

Let M be a vector space over a ring A of (left) operators. In the event that M has a basis over A, one may ask whether the number of basis elements is invariant. The author shows the answer to be affirmative if A is commutative, or if A can be embedded in a ring having the ascending and descending chain conditions for left ideals. An example where the answer is negative is provided by taking A to be the ring of linear transformations of an infinite dimensional space. Similar results were obtained by C. J. Everett [Bull. Amer. Math. Soc. **48**, 312–316 (1942); these Rev. **3**, 262], but neither paper contains the other.

I. Kaplansky (Cambridge, Mass.).

Dieudonné, Jean. Sur l'anneau des endomorphismes continus d'un espace normé. C. R. Acad. Sci. Paris **216**, 713–715 (1943). [MF 9992]

(I) Any automorphism of the ring of continuous linear operators of a real normed linear space in itself is necessarily an inner automorphism. (II) If the normed linear space is complex the automorphism is either an inner one or it is of the form $T\rightarrow A^{-1}TA$, where A is a one-to-one bicontinuous antilinear $(T(\lambda x+y)=\bar{\lambda}Tx+Ty)$ transformation of the space in itself. Of these two theorems, which are proved in this note, the first one had already been stated by Eidelheit [cf. Studia Math. **9**, 97–105 (1940); these Rev. **2**, 224] but the present author undoubtedly had no access to this result. The second one was found independently by B. H. Arnold [cf. Ann. of Math. (2) **45**, 24–49 (1944); these Rev. **5**, 147; see, in particular, section 10] in a paper submitted before but published after the appearance of the present note. The proof of Dieudonné is based on general properties of rings of linear operators, continuous with respect to different topologies. These properties were proved by the author in two papers [Ann. École Norm. (3) **59**, 107–139 (1942); Bull. Soc. Math. France **70**, 46–75 (1942)] which were not available to the reviewer. However, it seems that the methods of proof of Dieudonné and of Arnold should have many points of contact. *F. Bohnenblust.*

Snapper, Ernst. The resultant of a linear set. Amer. J. Math. **66**, 59–68 (1944). [MF 9939]

The theory, due to K. Hentzel and elaborated by Emmy Noether, of the resultant of an ideal of polynomials in n variables [Math. Ann. **88**, 53–79 (1923)] is here extended to a linear subset L of an m-dimensional vector space whose scalar domain consists of polynomials in n variables; that is, L is a linear subspace closed under scalar multiplication. The closure $\mathrm{Cl}(L)$ of L is defined to be the largest linear set whose scalar multiples lie in L. The principal result is that a polynomial resultant ρ is associated with every linear set L; ρ vanishes precisely for the common roots of the polynomials in the ideal $L/\mathrm{Cl}(L)$. The entire theory of Hentzelt is carried over to the present situation after being suitably modified. *E. R. Lorch* (New York, N. Y.).

Marrot, Raymond. Extension d'un théorème de Perron. C. R. Acad. Sci. Paris **217**, 165–167 (1943). [MF 10641]

The author proves that, if A is a completely continuous linear operator in Hilbert space and its matrix $\|a_{ij}\|$ relative to some orthogonal basis consists exclusively of positive numbers $a_{ij}>0$, then (1) there exists a $\lambda_0>0$ such that $Af=\lambda_0 f$, $f\neq 0$; (2) $Ag=\lambda g$ implies $\lambda_0>|\lambda|$; (3) the proper value λ_0 is simple. The finite dimensional case is due to O. Perron. Also established is the following. Given a continuous kernel $K(x, y)$ defined over a finite rectangle and satisfying $0<m\leqq K(x, y)\leqq M$, then K possesses a positive characteristic value of minimum modulus. *E. R. Lorch.*

Julia, Gaston. Sur l'adjoint de l'opérateur linéaire non borné défini par une matrice. C. R. Acad. Sci. Paris **216**, 853–856 (1943). [MF 10656]

The author considers a sequence of vectors A_n in a Hilbert space H, which has a dual sequence B_n, that is, $(A_i, B_j)=\delta_{ij}$ such that the closed linear manifolds determined by the sets $[A_n]$ and $[B_n]$ are identical with the space H. He is interested in the extent of the domain of existence $D_{A^{**}}$ of the adjoint A^{**} of A^* in D_{A^*}, the domain of A^*, where $A^*x=\sum e_n(x, A_n)$, the e_n forming an orthonormal basis for H. It is shown that $D_{A^{**}}$ contains all vectors $T=\sum t_i e_i$ for which the lower limit of $\|\sum^n t_i A_i\|<\infty$ and is contained in the set D of all T for which the upper limit of $\|\sum^n t_i A_{ni}\|<\infty$, where A_{ni} is the projection of A_i on the closed linear manifold determined by $B_1, B_2, \cdots, B_n$. If $A^*(D_{A^*})=H$, that is, if A^* is the inverse of a bounded operator, the classes $D_{A^{**}}$ and D are identical.

H. H. Goldstine (Aberdeen, Md.).

Wavre, R. L'itération directe des opérateurs hermitiens. Comment. Math. Helv. **16**, 65–72 (1944).

Let $x'=Ax$ be a Hermitian transformation in a space E (Hilbert space or a generalization) with inner product (x, y). An $x\epsilon E$ is normal if $\|x\|\equiv(x, x)=1$. Let $x_0\epsilon E$ and let $l_1, l_2, \cdots$ and $x_1, x_2, \cdots$ be positive constants and normalized elements of E such that $A^p x_0=l_1 l_2\cdots l_p x_p$ for each

$p=1, 2, \cdots$. From

$$l_1^2=(l_1x_1, l_1x_1)=(Ax_0, Ax_0)=(x_0, A^2x_0)$$
$$=l_1l_2(x_0, x_2)\leqq l_1l_2\|x_0\|\|x_2\|=l_1l_2$$

follows $l_1\leqq l_2$. Similarly $l_1\leqq l_2\leqq l_3\leqq\cdots$. Let $l=l(x_0)=\lim l_p$ and $\bar\omega=\bar\omega(x_0)=(l_1/l)(l_2/l)\cdots$. If $\bar\omega\neq 0$, the sequences x_{2p} and x_{2p+1} converge strongly to solutions of $A^2x=l^2x$; if $\bar\omega=0$, the sequences converge weakly to 0. If A is bounded, $\sum\|x_{p+2}-x_p\|^2<\infty$. If $l(x)>l(y)$, or $l(x)=l(y)$ while $\bar\omega(x)\neq 0$ and $\omega(y)=0$, then x has greater rank than y; denote this by $x\supset y$. For each x and n, the ranks of x and A^nx are equal. If $x\supset y$, then $(x_{2p}, y)\to 0$ and $\|A^{2p}y\|/\|A^{2p}x\|\to 0$ as $p\to\infty$. Ranks of sets are compared. The functions $l(x)$ and $\bar\omega(x)$ are respectively lower and upper semicontinuous. Points x_0 transformed to 0 by polynomials in A, projections and spectra are discussed briefly. *R. P. Agnew.*

Wavre, R. L'itération directe des operatéurs hermitiens et deux théories qui en dépendent. Comment. Math. Helv. 15, 299–317 (1943).

The author derives in a neat way many of the well-known properties of Hermitian operators in complex Hilbert space by studying the sequence of normalized iterates of an arbitrary element of the space under such an operator. The results are applied to the theory of integral equations with symmetric kernel and to the solutions of systems of linear equations in infinitely many unknowns.

Given an operator A and an element x_0, the sequences x_r of elements and l_r of nonnegative real numbers are determined by the relations: $A^rx_0=l_1\cdots l_rx_r$, where $\|x_r\|=1$ and $l_r=\|Ax_{r-1}\|$. It follows that $l_r\leqq l_{r+1}$, and the sequence l_r approaches a limit l. The properties of the operator A are related to the behavior of the functional $\omega_r(x_0)$ defined by the product: $\omega_r(x_0)=(l_r/l)(l_{r+1}/l)\cdots$. If $\omega_1(x_0)$ is not 0, the operator A is called regular. A regular operator is bounded, and a completely continuous operator is regular. It is shown that the even normalized iterates x_{2p} of a regular Hermitian operator A converge strongly to a characteristic element of A^2. Other results obtained are the spectral theorem for regular operators, and the Hilbert-Schmidt theorem. It is proved that the spectrum of a completely continuous operator has no accumulation points except 0. *O. Frink* (State College, Pa.).

Sobczyk, Andrew. On the extension of linear transformations. Trans. Amer. Math. Soc. 55, 153–169 (1944). [MF 10213]

This paper is concerned with the extensions of a linear transformation U, for which the range is permitted to increase. Here "linear" implies continuous. The author shows that, if U is one-to-one and has a complete domain X and if one can find an extension with a complete domain Z, with the same range as U, then there is a bounded projection of Z onto X. On the other hand, if one permits the range space to increase, one can extend the domain from any linear X to any linear Z, including X, without increasing the bound of the transformation. However, the new range may not be within any given space, which includes the old range. This result is compared with an analogous result of R. S. Phillips [Trans. Amer. Math. Soc. 48, 516–541 (1940); these Rev. 2, 318].

If the transformation is an isomorphism, it is possible to extend the range in a prescribed manner, retaining the isomorphism character, and similarly if U is one-to-one. If, however, one wishes to preserve the one-to-one character of U and to increase the domain in a prescribed way, the old domain must be closed in the new. Transformations are classified accordingly as their range is complete or not and accordingly as their set of zeros has a projection or not. The author presents the result that a U with an incomplete range can always be extended to one with a complete range. Essentially, this result is obtained by forming $X\oplus L_c$, where L_c is the completion of the range, and defining the extension U' by $U'(x, y)=Ux+y$.

The method used by the author is based on the notion of "convex functional," a real valued function $f(x)$ of the elements x of the space, such that $f(x)\geqq 0$, $f(tx)=|t|f(x)$ and $f(x_1+x_2)\leqq f(x_1)+f(x_2)$. Such a convex functional is always associated with a linear transformation U such that $f(x)=|U(x)|$ and also with a set of linear functionals on the range of $U(x)$. The extension of these latter by the Hahn-Banach theorem is used to extend f and hence U. These concepts form a most interesting combination.

The author also states a number of problems which he hopes "to deal with successfully in another paper." At least one question he asks can be immediately answered. "When X' and X'' are complementary closed subspaces, corresponding to a bounded projection P in a Banach space X, can there ever exist a closed subspace R such that R intersects X' and X'' only at the origin and PR is an incomplete subspace?" The answer is yes. Let X' and X'' be any two spaces and $X=X'\oplus X''$ and P the obvious projection of X on X'. Let R be the graph of a closed unbounded distributive transformation from X' to X'', which has an inverse. Then PR will be the domain of this transformation. *F. J. Murray* (New York, N. Y.).

Neumark, M. Positive definite operator functions on a commutative group. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 7, 237–244 (1943). (Russian. English summary) [MF 10833]

The author investigates the representation

$$f(x)=\int_{-\infty}^{\infty}e^{ix\alpha}d\varphi(\alpha), \qquad d\varphi(\alpha)\geqq 0$$

(and its generalization to Abelian groups) of positive definite functions in the case in which the values of f and φ are not numbers but bounded operators in Hilbert space. It may have escaped the author that the topic has been discussed before by the reviewer [S.-B. Preuss. Akad. Wiss. 1933, 371–376]. *S. Bochner* (Princeton, N. J.).

TOPOLOGY

Ribeiro, Hugo. Corrections à la note "Sur les espaces à métrique faible" (Port. Math., Vol. 4, (1943) Fasc. 1, p. 21–40). Portugaliae Math. 4, 65–68 (1943). [MF 10549]

Corrects misprints and a faulty proof contained in the note quoted in the title. Cf. these Rev. 5, 149. *O. Frink* (State College, Pa.).

Hopf, Heinz. Enden offener Räume und unendliche diskontinuierliche Gruppen. Comment. Math. Helv. 16, 81–100 (1944).

The universal covering space R (in fact, any regular covering space R) of a compact space is in general noncompact; it admits a discontinuous group G of transformations of itself. One may choose a compact subset whose

transforms cover R. The main problem of the paper is to find conditions that a noncompact space R must satisfy in order that it may admit such a group of transformations. The main theorem is that R must have either one or two or a Cantor set of ends (=prime ends). There are simple examples of each case. The number of ends depends on G only; in fact, the ends can be described directly in terms of G. However, the author does not know any purely algebraic theory of ends.

The definition of ends is given following Freudenthal [Math. Z. **33**, 692–713 (1931); see also Ann. of Math. (2) **43**, 261–279 (1942); these Rev. **3**, 315]; in fact, the proof of the main theorem leans on Freudenthal's paper. An end of R is determined, abstractly, by a sequence $G_1 \supset G_2 \supset \cdots$ of connected open subsets of R, each with a compact boundary and with no common point. The sequences $\{G_i\}$ and $\{G_i'\}$ determine the same end if each G_i contains a G_j'. An end may also be defined by a divergent sequence of points of R. If we add the ends of R to R, a compact space $\bar{R}$ is formed. The set G of transformations need not form a group, but it must be "strongly discontinuous": each two points x, x' of R must be in neighborhoods U, U' such that $f(U)$ has points in U' for at most a finite number of transformations f in G.

The principle facts used in proving the main theorem are the following. (1) Suppose $f_1(x)$, $f_2(x)$, $\cdots$ is a sequence defining the end E (and hence converging to E in $\bar{R}$). Then all compact sets M in R converge under $\{f_i\}$ to E: for each neighborhood U of E in $\bar{R}$ there is an n_0 such that $f_n(M) \subset U$ for $n \geqq n_0$. (2) To each end corresponds such a sequence $\{f_i\}$. (3) If R has at least three ends, then each end has, in an arbitrary neighborhood U of itself, another end. For the proof, $Q = R - U$ is taken connected. Neighborhoods H_i of ends E_i with compact boundaries K_i ($i = 1, 2, 3$) are chosen, and f is taken so that $f(K_1 + K_2 + K_3) \subset U$. The extension $\bar{f}$ of f through $\bar{R}$ then carries at least two of the E_i into U; each of $f(E_i)$ is an end. For further results, we refer to the full paper. *H. Whitney* (Cambridge, Mass.).

Lusternik, L. Ring of intersections in a certain functional space. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 59–61 (1943). [MF 8687]

Lusternik, L. Families of arcs with common end points on the sphere. C. R. (Doklady) Acad. Sci. URSS (N.S.) **39**, 88–90 (1943). [MF 9720]

The so-called "calculus of variations in the large" reduces many variation problems to the study of topological properties of suitable functional spaces. Hence it is of great importance to have methods for the effective computation of topological invariants of such spaces. In the first paper the author outlines a method for the computation of the cohomology groups and of their intersections. The functional space considered is that of rectifiable arcs joining two points a,b on the l-sphere S_l (for the sake of simplicity it is assumed that $l=2$), but the method could be applied in other similarly defined spaces. The other paper deals with the homology groups. As an application the author proves that, if $\bar{S}_2$ is a differentiable surface of genus 0, $\bar{R}$ the space of rectifiable arcs joining two points $a,b \in \bar{S}_2$, and K is a subset of R containing an n-dimensional cycle that does not bound in R, then K contains curves passing through any n points of S_2. *S. Eilenberg* (Ann Arbor, Mich.).

Lusternik, L. On dimensionality of critical sets. C. R. (Doklady) Acad. Sci. URSS (N.S.) **39**, 338 (1943). [MF 10755]

Given a twice differentiable function f defined on a twice differentiable manifold and given an isolated critical value c, the author considers the critical set M on the hypersurface $f=c$. A sufficient condition is given for M to contain an unbounding cycle of dimension r. *S. Eilenberg.*

Cairns, Stewart S. Isotopic deformations of geodesic complexes on the 2-sphere and on the plane. Ann. of Math. (2) **45**, 207–217 (1944). [MF 10263]

A simplicial complex on the 2-sphere is called geodesic if each of its 1-cells is an arc of a great circle shorter than a semi-circle and each of its 2-cells is a spherical triangle less in area than a hemisphere. The author is concerned with the isotropic deformations of geodesic complexes which remain geodesic throughout the deformation. The existence of such an isotopic deformation between two geodesic complexes is proved to be a necessary and sufficient condition that the isomorphism between the two geodesic complexes can be extended into an orientation-preserving self-homeomorphism of the 2-sphere. A similar theorem holds on the plane.

S. Chern (Princeton, N. J.).

Cairns, Stewart S. Introduction of a Riemannian geometry on a triangulable 4-manifold. Ann. of Math. (2) **45**, 218–219 (1944). [MF 10264]

According to results of an earlier paper [Ann. of Math. (2) **41**, 796–808 (1940); these Rev. **2**, 71] the isotopic deformation theorem on a 2-sphere [cf. the preceding review] has as consequence the following theorem. If M is a triangulable m-manifold, $m \leqq 4$, there exists an analytic Riemannian manifold homeomorphic to M. *S. Chern.*

Cairns, S. S. Deformations of plane rectilinear complexes. Amer. Math. Monthly **51**, 247–252 (1944). [MF 10562]

An elementary proof is given of a particular case of the isotopic deformation theorem in the plane [cf. the preceding review], namely, the case that the isomorphic complexes have three common vertices A_1, A_2, A_3 such that each of them exactly covers the triangle $A_1A_2A_3$ and that all other vertices of both complexes lie in the interior of the triangle $A_1A_2A_3$. *S. Chern* (Princeton, N. J.).

Whitney, Hassler. The self-intersections of a smooth n-manifold in $2n$-space. Ann. of Math. (2) **45**, 220–246 (1944). [MF 10265]

A mapping f of a manifold M^n into Euclidean space E^{2n} is regular if, at each point of M^n, n independent vectors are carried into n independent vectors. At each point of the image $f(M)$ consider the n-dimensional tangent plane. A double point of $f(M)$ is called a regular self-intersection if the two tangent planes there have just the one point in common. If the mapping has only regular self-intersections, and no triple points, it is called completely regular. The paper deals largely with a number I_f, called the intersection number of f (assumed completely regular); I_f is the algebraic number of self-intersections of $f(M)$ if n is even and M is orientable; otherwise I_f is determined by a reduction modulo 2 of the number of self-intersections without regard for sign (the sign is determined by considerations of orientation). If M is compact, I_f is invariant under regular deformations of f. If $n \geqq 3$, and if the total (not algebraic) number of self-intersections is greater than $|I_f|$, the

number may be reduced by 2 by a regular deformation. The main theorem is that any smooth n-manifold may be imbedded (mapped regularly and topologically) into E^{2n}.

A. E. Taylor (Los Angeles, Calif.).

Whitney, Hassler. The singularities of a smooth n-manifold in $(2n-1)$-space. Ann. of Math. (2) **45**, 247–293 (1944). [MF 10266]

The limit set of a mapping f of R into R' is the set of points q in R' such that, for some sequence $\{p_k\}$ in R, $f(p_k)\rightarrow q$, while $\{p_k\}$ has no limit point in R. The mapping is proper if the intersection of $f(R)$ and the limit set of the mapping is void. A mapping of a manifold is called an immersion if the mapping is proper and regular [see the preceding review for the definition of regular]. Such a mapping is locally one-to-one, but there may be self-intersections. A point at which a mapping is not regular is called a singularity. Part I of the paper is taken up with a study of singularities and self-intersections under mappings of M^n into E^{2n-1} [there is a misprint in theorem 2: read E^{2n-1} instead of E^{n-1}]. All singularities are reduced to a single type by slight alterations of the mapping to make it "semi-regular." An index of the algebraic number of singular points (reduced modulo 2 if n is even) is obtained by examining the mapping at the boundary of M. The principal theorem of the paper, in part II, states that, if an n-manifold ($n\geqq 2$) of class C^ν ($\nu\geqq 1$) is mapped continuously into E^{2n-1}, the mapping may be modified by an arbitrarily small amount in such a way as to give an immersion in E^{2n-1}, of class C^ν. If M^n is any compact "partial" manifold, it may be immersed in E^{2n-1} so that the mapping is an imbedding of some neighborhood of the boundary of M. Part III is concerned with the more complicated aspects of the theory begun in part I.

A. E. Taylor (Los Angeles, Calif.).

Thomas, T. Y. Lecture notes on Whitney's theory of the imbedding of differentiable manifolds in Euclidean space. Revista Ci., Lima **46**, 29–60 (1944). [MF 10760]

Part of Whitney's work on differentiable manifolds is segregated and clearly expounded. The principal results are that a connected compact separable n-dimensional manifold of class C^r ($r\geqq 1$) admits (1) a Riemannian metric of class C^{r-1} and (2) a regular topological imbedding of class C^r in a Euclidean $(2n+1)$-space. Clarity and simplicity are achieved by confining attention to the arguments needed for the establishment of these particular basic theorems and by introducing certain new techniques. The presentation here given should, aside from its intrinsic value, be useful to one who wishes to study some of the stronger, more general results to be found in Whitney's "Differentiable Manifolds" [Ann. of Math. (2) **37**, 645–680 (1936)].

S. S. Cairns.

Whyburn, G. T. Topological analog of the Weierstrass double series theorem. Bull. Amer. Math. Soc. **50**, 242–245 (1944). [MF 10205]

It is shown that, although even in the case of mappings of a sphere into a sphere the limit of a uniformly convergent sequence of light interior mappings may not be light interior, it can however be factored into a monotone transformation followed by a light interior one. A general theorem is then proved, from which it follows that, if the sequence $f_n(A)=B$ of light interior mappings of a locally connected continuum A converges uniformly to the mapping $f(A)=B$, then f is uniquely factorable in the form f_2f_1, where f_1 is monotone and f_2 is light and interior.

R. L. Wilder (Ann Arbor, Mich.).

Smith, P. A. Permutable periodic transformations. Proc. Nat. Acad. Sci. U. S. A. **30**, 105–108 (1944). [MF 10744]

The author presents here the first significant results in the study of the topology of finite noncyclic groups of transformations. He shows that the concept of special homology which he has applied with success in the cyclic case can be extended in a useful manner to the noncyclic case. The proofs given here are for the case of simplicial transformations and for the method of passing to the topological case the reader is referred to the author's article "Fixed Points of Periodic Transformations," which appears as appendix B in Lefschetz's "Algebraic Topology". [Amer. Math. Soc. Colloquium Publ., v. 27, New York, 1942; these Rev. **4**, 84] and which contains a summary of the author's work on the cyclic case.

Let M be a homological n-sphere over the integers modulo p. In addition M is required to satisfy certain local conditions which would be met if M were locally Euclidean. If t and s are permutable transformations of prime period p operating in M and if their fixed point sets are identical (possibly null), then s is a power of t. There cannot operate in M a commutative group of fixed point free transformations which is of order p^a, $a>1$, and of type $(p,\cdots,p)$. If a commutative transformation group of order p^k and type $(p,\cdots,p)$ operates on M, then $k\leqq(n+1)/2$. The author has remarked in conversation that it is a corollary of his results that, if G is a compact Lie group of transformations of an n-sphere or Euclidean n-space, then the rank of G is less than or equal to $(n+1)/2$.

D. Montgomery.

Schweigert, G. E. Fixed elements and periodic types for homeomorphisms on s.l.c. continua. Amer. J. Math. **66**, 229–244 (1944). [MF 10569]

This paper considers a semi-locally-connected continuum S and a homeomorphism $T(S)=S$. A special decomposition $S(p)$ of S is a decomposition $S=H+K$, where H and K are continua having precisely the point p in common, where p is a cutpoint of S, $H-p$ is a component of $S-p$, and H contains a fixed cyclic element E of S under T. By a bordered A-set B the author means that there exists a point x and a true cyclic element E of S such that $E\cdot B=x$. Using these concepts he shows: if $T^n(K)\cdot K\neq 0$ and $n\neq 0$ is the least integer giving this intersection, then either (a) p is of period n or (b) the period of p is infinite and there exists in $H-p$ an unbordered nonvacuous T^n-invariant nodal set P which contains every T^n-invariant set in H. In case (b) there exist two T^n-fixed points x and z such that the cyclic chain $C(x,z)$ with endpoints x and z is the least T^n-invariant A-set which contains the T^n-orbit of p. If $T^n(K)\cdot K=0$ for all $n\neq 0$, then the period of p is infinite and there exists in H an invariant nonvacuous A-set Q having for its set-theoretic boundary the closure of the orbit of p, under T.

It is shown that for any homeomorphism $T(S)=S$ each of the following nine conditions is necessary and sufficient to insure the other eight. (1) T is elementwise periodic on the cyclic elements of $S-L$, where L is the set of all endpoints of S, (2, 3, 4) T is componentwise periodic (that is, if A is a cutpoint, an endpoint or a nondegenerate A-set of S then every component of $S-A$ has a finite period under T) and, for every positive integer n, T^n has one of the first three properties of Ayres [cf. chapter 12 in G. T. Whyburn, Analytic Topology, Amer. Math. Soc. Colloquium Publ., v. 28, New York, 1942; these Rev. **4**, 86]. (5) T is componentwise periodic and, for every n, T^n has the first

property of Whyburn, that is, if X is an invariant A-set in S and N is an invariant node of X such that $N \neq X$, then there exists a fixed cutpoint x of X. (6) T is componentwise periodic and, for every positive integer n, T^n has the second property of Whyburn, that is, for every divison $S=H+K$ of S into continua H and K having precisely the point p in common, and such that neither $H\cdot T(H)$ nor $K\cdot T(K)$ is vacuous, it follows that $T(p)=p$. (7) If M is any subset of $S-L$ which is a sum of cyclic elements, then $T(M)\subset M$ implies $T(M)=M$. (8) T admits an expanding approximation to $S-L$, that is, if $I(n)$ denotes the sum of all cyclic elements of S which are invariant under $T^n(S)=S$, then there exists a sequence of integers $n_1, n_2, \cdots$, such that (a) $I(n_1)\subset I(n_2)\subset\cdots$, (b) $\sum I(n_i)\supset S-L$, (c) for any $\epsilon>0$ there exists an i such that each component of $S-I(n_i)$ is of diameter less than ϵ. (9) T is componentwise periodic and each unbordered T^n-invariant A-set A in S admits a contracting T^n-approximation which preserves interiority at A, that is, for every A-set C such that $A\subset \operatorname{Int} C$ there is a third T^n-invariant A-set B such that $A\subset \operatorname{Int} B\subset C$.

D. W. Hall (College Park, Md.).

Wecken, Franz. Fixpunktklassen. II. Homotopieinvarianten der Fixpunkttheorie. Math. Ann. 118, 216–234 (1941). [MF 10702]

[Part I appeared in Math. Ann. 117, 659–671 (1941); cf. these Rev. 3, 140.] Let $\mathfrak{P}$ be a complex, p a point in $\mathfrak{P}$, f a mapping $f\mathfrak{P}\subset\mathfrak{P}$. On choosing a definite path $\mathfrak{C}$ from p to $f(p)$ there is determined a homomorphism $H\Gamma\subset\Gamma$, Γ being the fundamental group of $\mathfrak{P}$. Call elements α, β of Γ H-conjugate if $\beta=(H\gamma)\alpha\gamma^{-1}$ for suitable γ in Γ. It is shown that the fixed-point classes of f can, in a natural manner, be put into one-one correspondence with the classes $\mathfrak{p}_1^H$, $\mathfrak{p}_2^H$, $\cdots$ of H-conjugate elements (provided certain conventions are made about the empty classes). In this way there is associated with each $\mathfrak{p}_i^H$ an integer v_i, namely, the algebraic fixed-point number of the corresponding fixed-point class. At most a finite number of v_i's are nonzero. It is shown that the expression $\sum v_i \mathfrak{p}_i^H$ is identical with a certain invariant (Reidemeister) of the automorphism which $\{f, \mathfrak{C}\}$ determines in the homotopy ring of chains of the universal covering of $\mathfrak{P}$.

P. A. Smith (New York, N. Y.).

Wecken, Franz. Fixpunktklassen. III. Mindestzahlen von Fixpunkten. Math. Ann. 118, 544–577 (1942). [MF 10714]

[Cf. the preceding review.] It is shown that the minimum number of fixed points of a mapping $f(\mathfrak{P})\subset\mathfrak{P}$ equals the number of essential fixed-point classes in the following cases: (1) $\mathfrak{P}$ is a 2-dimensionally connected finite ~~2-dimensional~~ complex and f is of the class of the identity; (2) $\mathfrak{P}$ is a 3-dimensionally connected finite ~~3-dimensional~~ complex such that any path joining two regular points of $\mathfrak{P}$ is homotopic to a path lying in an open 3-dimensional subset of $\mathfrak{P}$. The result in case (1) follows from the theorem that a finite 2-dimensionally connected ~~finite 2~~ complex $\mathfrak{P}$ of Euler characteristic χ admits a deformation without fixed points if $\chi=0$, with a single fixed point of index χ if $\chi\neq 0$.

P. A. Smith (New York, N. Y.).

Heawood, P. J. Note on a correction in a paper on map-congruences. J. London Math. Soc. 18, 160–167 (1943). [MF 10368]

In a previous paper [Proc. London Math. Soc. (2) 40, 189–202 (1935)] the author conjectured that, if two adjacent regions of a complete map were left out of account, the congruences (mod 3) corresponding to the remaining regions could always be solved no matter what constants were assigned to these remaining regions. If this conjecture were true, the four color theorem would follow by taking the constants all zero and applying the author's "doctrine of residuals," explained in the same paper. In part I of the present paper the author shows by an example that the conjecture was wrong, but he offers two weaker conjectures of the same type which would, if true, also imply the four color theorem.

In part II the author gives the congruence-theoretic interpretation of the intuitive supposition that, in coloring all the regions within an n-ring, the possible colorings of the n-ring tend to be more numerous the greater the number of regions within the n-ring. We assume a normal map with no rings of less than five regions. He gives special attention to the case $n=5$ in connection with our desire to arrive finally at a 5-ring and a single "exterior" region. It is, of course, crucial to show that this final 5-ring can always be colored in three colors. The author gives the corresponding crucial requirement in terms of residuals.

D. C. Lewis.

MATHEMATICAL PHYSICS

Synge, J. L. Focal properties of optical and electromagnetic systems. Amer. Math. Monthly 51, 185–200 (1944). [MF 10255]

The paper deals with the first order laws of electron optics, the so-called laws of Gaussian optics. A new slant is given to the derivation of the well-known facts by utilizing the similar form of the Cauchy-Riemann differential equations and the differential equations of optics. The real differential equations of optics, in the case of rotation symmetry, can thus be transformed into a single differential equation for a complex vector. The author proves that in the case of a statical electromagnetic field there always exists an object which has a real image.

M. Herzberger.

Kousnetsov, E. S. Theory of non-horizontal visibility. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] 1943, 247–336 (1943). (Russian. English summary) [MF 10726]

[Reviewed from the English summary.] This paper gives a general theory of the range of vision through the atmosphere in any direction, in contrast to existing theories, which deal chiefly with vision in the horizontal. Starting from a generalized form of the extinction equation, a system of integral equations is set up, which can be solved under certain assumptions concerning the laws of scattering. The solutions of these equations are then applied to the calculation of the visual range. The last part of the paper deals with the solutions that correspond to certain special laws of scattering. As far as can be determined from the English summary, there is no attempt to deal with the problems presented by a stratified distribution of atmospheric aerosols, so often found in practice. Nevertheless, the paper is a noteworthy attempt to deal with the formidable difficulties of this new subject.

W. E. K. Middleton.

Esclangon, Ernest. Sur les réfractions géodésiques. C. R. Acad. Sci. Paris 216, 137–139 (1943). [MF 10005]

The author expresses the refraction between an observation point A and another B in terms of the indices of refraction at the points, the zenith distance of the ray at A,

the angle between the ray direction at B and the vertical at A, and distances from the earth's center. The result is stated in terms of the erf function. *D. G. Bourgin.*

Garfinkel, Boris. An investigation in the theory of astronomical refraction. Astr. J. 50, 169–179 (1944). [MF 10091]

This paper presents a theory of the astronomical refraction in an atmosphere in which the temperature is a linear function of the "dynamical height." Tables and formulae are given by which the refraction may be computed for objects at zenith distances up to 116°, using the values of pressure and temperature at the elevation of the observer, and a mean value of the temperature gradient in the troposphere. Corrections, important only for objects at zenith distances greater than 80°, arise from certain approximations in the theory; these are evaluated. A correction for the presence of an isothermal stratosphere is computed for various zenith distances and heights of observer (up to 10 km); and finally the author gives a scheme by means of which the observed temperature gradient in the troposphere may be introduced to yield a fourth correction. While the variation of the temperature gradient is the principal source of uncertainty in the computation of the refraction for objects near the horizon, it is of little importance for practical measurements when the zenith distance is less than 80°.

There is some typographical confusion between μ and u on page 169. *W. E. K. Middleton* (Toronto, Ont.).

Grünberg, G. A. On the distribution of electricity on thin unclosed conducting layers. Acad. Sci. USSR. J. Phys. 7, 93–98 (1943). [MF 9965]

The distribution of electricity on thin shell segments is studied and integral formulas are derived for the charge densities on both sides of the shell. *H. Poritsky.*

Parodi, Maurice. Propagation des courants sinusoïdaux dans une ligne quelconque. C. R. Acad. Sci. Paris 216, 876–878 (1943). [MF 10660]

A current and voltage of fixed frequency $\omega/2\pi$ are carried by a long line with "constants" L, R, C and G which are all functions of x, where x is distance down the line. Let $j\omega L+R$ be denoted by $\alpha(x)$ and $j\omega C+G$ by $\beta(x)$. The author determines $A(x)$, $B(x)$, $C(x)$ and $D(x)$ as infinite sums of iterated integrals involving $\alpha(x)$ and $\beta(x)$. If α and β are constant, as in the usual case, A, B, C and D are the usual hyperbolic sines and cosines. The usual relationships $V=AV_0+BI_0$ and $I=CV_0+DI_0$ then express V and I at any point x in terms of V_0 and I_0, the voltage and current vectors at $x=0$. The author's proof, although he does not seem to realize it, simply involves solving the usual pair of long line equations by Picard successive approximations. *N. Levinson* (Cambridge, Mass.).

Consoli, Terenzio. Fréquences principales des circuits en Π et en T. Rev. Fac. Sci. Univ. Istanbul. (A) 8, 39–61 (1943). (French. Turkish summary) [MF 10184]

This paper deals with the evaluation of the determinants occurring in filter networks containing a finite number of Π- and T-shaped branches. One of the determinants investigated, $D_n(x)$, consists of the variable x in the main diagonal, of the constant a in the two adjacent diagonals and of zero terms elsewhere. Its expansions in powers of x are given in detail and its roots obtained. A similar treatment is given to the determinant Δ_n, which is similar to D_n except for different upper left and lower right terms, bx. Graphical methods for the treatment of the filter problems are also indicated. *H. Poritsky* (Schenectady, N. Y.).

De Simoni, Franco. Teoria elementare dei risonatori sferici cavi eccitati da un dipolo hertziano. Alta Frequenza 12, 163–182 (1943). [MF 10357]

In order to obtain the solution for the electromagnetic field of a spherical resonator excited by an electron beam through a small gap in the center of the sphere, the author substitutes the much simpler problem of an infinitesimal Hertzian dipole in the center of an otherwise completely empty spherical metallic shell. Using the Hertzian vector potential and assuming complete axial symmetry, the field interior to the spherical shell becomes the superposition of the classical dipole field and a "reflected" field, while the electromagnetic field in the metal of the shell is obviously so rapidly attenuated that a "reflection" term from the outer surface of the metal shell can be disregarded. Satisfying the conditions of continuity of the tangential field-components on the inner surface of the metallic shell, a very complicated frequency equation is derived which defines the resonant modes of the system. By suitable approximations the condition for the lowest resonant frequency becomes $\tan x=x/(1-x^2)$, which is solved graphically and gives $x=2.74=(w_0a/c)$, where w_0 is the lowest resonant angular frequency, a the inner radius of the metallic shell and c the velocity of light. From this follows the longest resonant wave length as $l_0=2.29a$, as well as an approximate value for the associated attenuation constant.

Though the principal analysis is carried through with exponential and trigonometric functions for the Hertzian vector potential, the author shows in a later section the equivalent solution in terms of Bessel functions of order $\frac{1}{2}$ and demonstrates the identity of the two types of solutions for the individual field vector components. *E. Weber.*

Page, Leigh. The electrical oscillations of a prolate spheroid. II. Prolate spheroidal wave functions. Phys. Rev. (2) 65, 98–110 (1944). [MF 10087]

Page, Leigh. The electrical oscillations of a prolate spheroid. III. The antenna problem. Phys. Rev. (2) 65, 111–117 (1944). [MF 10088]

Paper I of this series by L. Page and N. I. Adams [Phys. Rev. (2) 53, 819–831 (1938)] contained a discussion of the free oscillations of a prolate spheroidal conductor as well as the forced oscillations due to a plane wave parallel to the long axis of the spheroid. These latter oscillations, which are needed for applications to the straight wire antenna, are more exactly and simply treated in part II. The variables of the vector wave equation in spheroidal coordinates are separated, resulting in a pair of ordinary differential equations of the type

$$\frac{d}{dx}\left[z\frac{dy}{dx}\right]-\left\{\frac{m^2}{z}-\alpha-\epsilon^2\right\}y=0, \qquad z=1-x^2,$$

where m is a positive integer. Solutions y exist only for a discrete set of characteristic values α_{lm} of α. The corresponding characteristic functions $y_{l,m}$ are best expressed as a rapidly converging series of Legendre functions $P_l^{(m)}(x)$ in even powers of ϵ. Characteristic values $\alpha_{l,m}$ and functions $y_{l,m}$ are tabulated for $(l, m)=(1, 1)$, $(2, 1)$, $(3, 1)$, $(4, 1)$, $(5, 1)$, $(2, 2)$ and $(3, 2)$. A discussion is made of the zeros of $\alpha_{l,m}$ for the case $m=1$.

Paper III makes applications of the results of paper II to the case of a finite straight wire antenna (a prolate ellipsoid of eccentricity nearly unity). First the free oscillations which have axial symmetry are treated as far as the third harmonic, together with the wave length and logarithmic decrement. Next, the problem of the forced oscillation mentioned above is discussed as regards the frequencies attained close to resonance. Only the first two resonances are considered, the first of these in considerable detail. Expressions are given for the current, radiation resistance and field components (in spheriodal coordinates) of the wave transmitted by the antenna at resonance. *D. H. Lehmer.*

Fierz, M. Zur Theorie der Kapillarschwingungen eines Flüssigkeitstropfens. Helvetica Phys. Acta 16, 365–370 (1943).

This paper consists of certain corrections to an earlier paper by S. Flügge [Ann. Physik (5) **39**, 373–387 (1941); these Rev. **3**, 318] on the capillary oscillations of a non-viscous charged liquid drop. In particular, it is shown that, contrary to Flügge's statements, spherically symmetric capillary oscillations are, when properly interpreted, characterized by angular momenta. Thus the oscillation described by the velocity potential

$$\beta_{l,m} r^l \mathfrak{P}_{l,m}(\cos \vartheta) \sin (m\varphi - \omega t)$$

and the equation of the surface

$$R(\vartheta, \varphi) - R_0 = \alpha_{l,m} \mathfrak{P}_{l,m}(\cos \vartheta) \cos (m\varphi - \omega t),$$

where the $\mathfrak{P}_{l,m}$'s are the normalized associated Legendre functions, has associated with it an angular momentum $J_z = mE/\omega$ about the z-axis, where E denotes the energy (kinetic+potential) in the drop. Furthermore, associated with this angular momentum the drop, when electrically charged, has also a magnetic moment $\mu_z = (Ze/2Mc)J_z$, where Ze denotes the charge and M the mass of the drop and c the velocity of light. This difference between the results of Fierz and Flügge arises from the fact that, while Fierz uses progressive wave solutions, Flügge bases his discussion on stationary wave solutions.

The rest of the paper is devoted to correcting certain oversights in Flügge's evaluation of the rate of radiation of electromagnetic energy by a charged liquid drop undergoing capillary oscillations of the type considered.

S. Chandrasekhar (Williams Bay, Wis.).

Podolsky, Boris and Kikuchi, Chihiro. A generalized electrodynamics. II. Quantum. Phys. Rev. (2) 65, 228–235 (1944). [MF 10277]

In an earlier paper by the first named author [Phys. Rev. (2) **62**, 68–71 (1942); these Rev. **4**, 31] a generalized set of equations for an electro-magnetic field was derived from a Lagrangian function involving derivatives of the field strengths. In this paper these equations are written in Hamiltonian form and quantized. General methods for writing the Hamiltonian form and for determining the energy tensor are given. The auxiliary conditions, which are different in the case here considered from those in the usual theory, are discussed in some detail. *A. H. Taub.*

Dirac, P. A. M. The physical interpretation of quantum mechanics. Proc. Roy. Soc. London. Ser. A. **180**, 1–40 (1942). [MF 10818]

This is the author's Bakerian lecture for 1941. There are three mathematical appendices: (1) The action principle of classical electrodynamics. (2) Relativistic second quantization. (3) The transformation connecting the two forms of quantum electrodynamics. Cf. the following review.

Pauli, W. On Dirac's new method of field quantization. Rev. Modern Phys. **15**, 175–207 (1943). [MF 9282]

The method of field quantization proposed by Dirac [cf. the preceding review], which uses an indefinite metric in the Hilbert space of quantum states, is discussed in detail. The physical meaning of the mathematical formalism is not clear since it leads to statements such as the following: there is a negative probability that certain positive eigenvalues of an observable are realized. This formalism is used in conjunction with a limiting procedure called the λ-limiting process. The latter is a method, applicable to classical mechanics, which introduces a time-like vector λ^μ. It is assumed that two particles never approach to a space-time distance of the order $(\lambda^\mu\lambda_\mu)^{\frac{1}{2}}$ from one another. The limit is then taken as $\lambda^\mu \to 0$ subject to the restrictions that λ^μ is time-like. The convergence difficulties of quantized field theories may be overcome by using the indefinite metric in Hilbert space and this limiting process. New difficulties arise in connection with the theory of the generation and annihilation of pairs of particles with opposite electric charges. The author feels that these may be removed by replacing the limiting process by some purely quantum theoretical method.

The following problems are discussed: (1) the harmonic oscillator, (2) system of two oscillators with positive and negative energy, (3) uncharged particles with spin 0 and 1 and Bose statistics and (4) charged particles of spin 0. The paper contains a new discussion of the λ-process involving only a single time variable in the section entitled "The interaction of electrons with an electromagnetic field." In this section the roles that the two formalisms play in achieving convergence is discussed in detail.

The paper concludes with a discussion of the problems involved in the physical interpretation of the formalism. The introduction of a hypothetical world for which the collision processes are the same as in the actual world is discussed for the examples treated in the earlier parts of the paper and the arbitrariness of the rules for carrying over the results regarding the former to the latter is pointed out.

A. H. Taub (Princeton, N. J.).

Pauli, W. On applications of the λ-limiting process to the theory of the meson field. Phys. Rev. (2) 64, 332–344 (1943). [MF 9707]

The so-called λ-limiting process previously applied by Wentzel and Dirac to the case of the electromagnetic field is here applied to the problem of the interaction of nuclear particles with meson fields. By means of this limiting process the divergences of point source models are avoided without introducing a finite extension of the source and without destroying the relativistic invariance of the theory. The inertia constants of the degrees of freedom of the nuclear particles spin and isotopic spin are zero in this formulation of the theory.

It is shown that the pseudo-scalar and vector meson theories with a coupling constant f of the dimension of a length have the character of a weak coupling theory and that no stable isobars exist when $\mu f \ll 1$; μ is the reciprocal of the Compton wave-length of the meson.

The higher approximations of the interaction of the nuclear particles with the meson field proposed by Rosenfeld and Møller are discussed. The latter field consists of a pseudo-scalar and a vector field each with the same coupling

constant f. These interactions are expanded in powers of f and the method of successive canonical transformations due to Stueckelberg and Patry is used to show that the λ-process makes the coefficients of the higher powers of f finite as long as the nuclear particles can be supposed to be at rest. These are also shown to be small compared to the coefficient of f^2 when the distances between nuclear particles are sufficiently larger than f. *A. H. Taub.*

Wentzel, Gregor. Zur Vektormesontheorie. Helvetica Phys. Acta 16, 551–596 (1943).

Die symmetrische Vektortheorie mit dem allgemeinen, zweiparametrigen Kopplungsansatz (statische Näherung) wird für den Fall starker Kopplung untersucht. Die Rechnung verläuft verschieden, je nachdem ob das Verhältnis f/g der beiden Kopplungsparameter dem Betrage nach unterhalb oder oberhalb eines kritischen Wertes liegt. Im ersten Falle sind die Ergebnisse wesentlich dieselben wie beim speziellen Kopplungsansatz $f=0$ (keine Kopplung der longitudinalen Mesonen). Für beide Fälle werden abgeleitet: die Isobaren-Energie, der Mesonstreuquerschnitt und die statischen Kernkräfte für grosse Abstände. *Author's summary.*

Gupta, S. On the electromagnetic field and the self-energy of meson. Proc. Nat. Inst. Sci. India 9, 173–192 (1943). [MF 10173]

The interaction of mesons with spin 0 and 1 with the electromagnetic field is discussed, using the Duffin-Kemmer matrix formalism. The self-energy of the meson for the two values of the spin is calculated, following the method used by Weisskopf for the positron theory. The expression contains four terms, the first due to the longitudinal electric field, the second due to the transverse electric field, the third due to the magnetic field and the fourth due to the zero-point fluctuations. The results agree with those obtained earlier by Weisskopf for spin 0 and Richtmyer for spin 1. *S. Kusaka* (Northampton, Mass.).

Gupta, S. On the elastic scattering of the fast mesons. Proc. Nat. Inst. Sci. India 8, 369–375 (1942). [MF 10134]

The differential cross-sections for the elastic scattering of mesons with spin 1 and 0 by a Coulomb field are calculated by a method using the Duffin-Kemmer matrix formalism. The results agree with the expressions obtained earlier by Laporte and by Massey and Corben. *S. Kusaka.*

Gupta, S. and Majumdar, R. C. On the collision between meson and electron. Proc. Nat. Inst. Sci. India 8, 199–216 (1942). [MF 10135]

Calculations are given for the cross-sections for collision between an electron and a meson of spin 0, $\frac{1}{2}$ and 1 by the usual method of quantum electrodynamics. The results agree with those obtained earlier by Massey and Corben. The bearing of these calculations on the burst production by mesons is discussed. *S. Kusaka.*

Weinstock, Robert. Inelastic scattering of slow neutrons. Phys. Rev. (2) 65, 1–20 (1944). [MF 9879]

The scattering of a neutron is said to be inelastic (or elastic) according as the kinetic energy of the neutron does (or does not) change in the process. In the analytical treatment the chance is required that, as a result of the interaction between a crystal and a neutron, there will be a transition from a definite initial state described by the wave-function Φ_n and the momentum $\mathbf{p}$ to the neutron momentum $\mathbf{p}'$ and the crystalline state Φ_n'. The crystalline potential energy U is expressed in the form

$$U=\tfrac{1}{2}\sum_s \omega_s^2(\gamma_s^2+\mu_s^2),$$

where γ_s, μ_s are parameters which occur linearly in the expression for the instantaneous deviation from equilibrium of the σth nucleus. Expressions are given for the total wave function describing the thermal agitation of the crystal, the total wave function of the system neutron plus crystal without interaction and the wave function after interaction, the three expressions being in the form of products. Use is then made of Born's approximation and the resulting analysis involves complicated integrals. A final result is expressed in terms of the function

$$Q_1(z)=\int_0^z t\,dt/(e^t-1).$$

Elastic and inelastic scattering are treated successively in sections A and B and a special case is considered which leads to the conclusion that, for neutrons of velocity less than the sound velocity in the scattering substance, single phonon emission cannot occur at all in this special case. Approximate results for phonon absorption are also found. The results for the total inelastic cross-section are expressed with the aid of a reciprocal lattice vector $\boldsymbol{\tau}$ of magnitude τ and a summation is made over all suitable values of τ, whereas, in the special case just mentioned, τ had the value 0. A comparison is made with the X-ray problem and results are given for both high and low temperatures. The inelastic scattering cross-section is plotted against the temperature of the scattering crystal and a comparison is drawn between total and elastic scattering cross-sections, the temperature being used as abscissa. In the numerical results use is again made of the function $Q_1(z)$ and of a related integral $Q_0(z)$ which is equal to $\log[\exp(z)-1]$. Approximations for Q_0 and Q_1 valid for $z\ll 1$ are given. *H. Bateman.*

Hamilton, J. and Peng, H. W. On the production of mesons by light quanta and related processes. Proc. Roy. Irish Acad. Sect. A. 49, 197–224 (1944). [MF 9984]

This deals with the transformation of a photon into a meson during collision with a nucleon (that is, a proton or neutron), and the reverse process, on the basis of a general theory of radiation damping developed by Heitler and Peng [Proc. Cambridge Philos. Soc. 38, 296–312 (1942); see also papers by Heitler and Wilson, respectively, Proc. Cambridge Philos. Soc. 37, 291–300, 301–316 (1941); these Rev. 4, 95]. Transition probabilities are calculated in the form of meson cross-sections for both low and high energies, and for longitudinal, transverse and pseudoscalar mesons. Agreement with experimental values is not very close, but the cross-sections calculated are of the right order of magnitude and remain bounded for large values of the energy as they should.

Since the transitions discussed do not play a large role in cosmic ray phenomena, the emphasis of the paper is on the mathematical details and methods, which are applicable also to other processes. The calculations involve the solution of simultaneous nonhomogeneous integral equations of general Fredholm type by the method of reciprocal kernels, and expansions in series of Legendre functions. It is pointed out that the theory violates the "principle of detailed balance," according to which the transition probabilities for

inverse processes are equal, at least in the case where the spin is included in the description of a state. *O. Frink.*

Chang, T. S. The impulse-energy tensor of material particles. I. Mesons and electrons. Proc. Roy. Soc. London. Ser. A. **182**, 302–318 (1944). [MF 10171]

The author finds the most general form for the impulse-energy tensor T_{kl} of material particles interacting with electrons and neutrinos, such that T_{kl} is real, gauge-invariant and symmetric in the indices k and l. The material particles considered are scalar mesons, vector mesons and particles of spin 2 and $\frac{3}{2}$ as treated by Fierz and Pauli. The expressions obtained for T_{kl} involve the wave functions of the particles in question, together with their derivatives up to the second order. They are quite complicated and are not unique. To justify the interpretation of T_{kl} as the energy-momentum tensor, two checks are applied in the case of scalar and vector mesons. It is found that, for vanishing electric field, the expressions for $(i/c)T_{k4}$ reduce, except for a single term, to the form to be expected for the momentum density vector. Secondly, it is found that the integral over space of T_{44} is equal to the integral over space of the Hamiltonian H', when the latter is modified to make it gauge-invariant. *O. Frink* (State College, Pa.).

Destouches, Jean-Louis. Sur le principe de décomposition spectrale en mécanique ondulatoire. C. R. Acad. Sci. Paris **215**, 523–525 (1942). [MF 10187]

The author states without proof a theorem concerning a certain abstract generalization of wave mechanics called the general theory of previsions. His theorem is that a constant k which occurs as an exponent in this theory is arbitrary if all physical quantities in the system are simultaneously measurable. For convenience it may then be assumed that $k=2$. But if there are physical quantities not simultaneously measurable, the constant k is unique; the statement that $k=2$ then expresses an important physical law, the principle of spectral decomposition. The description of the theory of previsions given in this note is too brief to be readily intelligible. *O. Frink* (State College, Pa.).

Destouches, Jean-Louis, Steinberg, Jean-Louis et Viard, Jeannine. Sur une formule de cinétique opératorielle. C. R. Acad. Sci. Paris **217**, 131–133 (1943). [MF 10640]

It is pointed out that the vector identity

$$(\overrightarrow{OM}\times p)^2+(\overrightarrow{OM}\cdot p)^2=\overrightarrow{OM}^2\cdot p^2$$

corresponds in quantum mechanics to the relation

$$(\overrightarrow{OM}\times p)^2+(\overrightarrow{OM}\cdot p)^2=\overrightarrow{OM}^2\cdot p^2-(ih/2\pi)(\overrightarrow{OM}\cdot p)$$

if p is the vector operator corresponding to the momentum of the particle M. It is also shown that a similar formula holds for the center of gravity of a system of particles. Applications are considered in the setting up of the wave equation in cases where an integral of angular momentum is given a fixed value. *D. C. Lewis* (New York, N. Y.).

Houriet, A. Forces nucléaires de la théorie des paires. Helvetica Phys. Acta **16**, 529–550 (1943).

This is an extension of the work of Jauch [Helvetica Phys. Acta **15**, 175–191 (1942)] on the electron pair theory of nuclear forces. As in the previous work, the coupling between the electron field and the nucleons is assumed to be of a scalar type, and the calculation of the nuclear forces is carried out by the method developed by Wentzel [Helvetica Phys. Acta **15**, 111–126 (1942)]. The present paper deals with the case of strong coupling. The results are that, for $\mu r\ll 1/Ar\ll 1$, where μ is the reciprocal Compton wave length of the electron and $1/A$ the size of the nucleon, the potential between two nucleons at a distance r apart is given by $V(r)=-4/Ar^2$ and, for $\mu\gg 1/Ar$,

$$V(r)=-\frac{4e^{-2\mu r}}{A^2r^3(\pi\mu r)^{1/2}}.$$

Furthermore, for a system of Z nucleons contained in a cube of side a, the energy is

$$U_Z(a)\simeq -64\pi^{-1}\gamma^{-2}Za^{-2}A$$

if $Z^{1/3}/A\ll a\ll 1/(\mu A)^{1/2}$, and $U_Z(a)\simeq -ZA$ in the limit $a=0$. Thus the forces are attractive and have the saturation property and, from the nature of the assumptions made, they are independent of the charge and spin of the nucleons. *S. Kusaka* (Northampton, Mass.).

Petiau, Gérard. Sur la représentation des interactions corpusculaires par l'intermédiaire de la particule de spin 1. C. R. Acad. Sci. Paris **216**, 832–834 (1943). [MF 10655]

Petiau, Gérard. Sur la représentation d'interactions neutron-proton s'exerçant par l'intermédiaire de la particule de spin 2. C. R. Acad. Sci. Paris **217**, 103–104 (1943). [MF 10635]

In the first paper the system consisting of a particle of spin one and a particle of spin one-half is represented by a three index spinor. The equation satisfied by this spinor is written in the form: the time derivative of the spinor is equal to a linear operator times the spinor. This operator is composed of these terms: (1) the Dirac operator for the particle of spin one-half, (2) the analogue of the Dirac operator for the particle of spin one and (3) the interaction operator which is taken to be an invariant linear combination of the Dirac type of matrices associated with the particles of spin one and one-half multiplied by a delta function of the positions of the two particles. The matrix components of the latter operator are computed.

The method given in the first paper is used to discuss the system composed of a neutron-proton (taken to be a particle of spin one-half represented by a four component spinor) and a particle of spin two. The total system is represented by a five index spinor. The interaction energy is allowed to have differential operators in this case. *A. H. Taub* (Princeton, N. J.).

Koppe, Heinz. Eine Näherungsmethode zur Berechnung der magnetischen Suszeptibilität. Z. Phys. **121**, 614–628 (1943). [MF 9977]

The characteristic functions of the wave equation for a hydrogen atom in a constant magnetic field are obtained by a perturbation method and the results applied to the two-electron problem. Thus a correction term to the usual mean-value theory of atomic susceptibility is obtained, and the author shows that it leads to better agreement with the available experimental values. A discussion of the variation of the susceptibility with field strength is also included. *M. C. Gray* (New York, N. Y.).

v. Laue, M. Nachtrag zu meiner Arbeit: "Eine Ausgestaltung der Londonschen Theorie der Supraleitung." Ann. Physik (5) **43**, 223–224 (1943). [MF 9927]

Correction of an erroneous numeric factor in the expression for the energy momentum tensor given in Ann. Physik (5) **42**, 65–83 (1942); cf. these Rev. **5**, 162. *F. London* (Durham, N. C.).

Statistical Mechanics

Khintchine, A. Sur le problème ergodique de la mécanique quantique. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 7, 167–184 (1943). (Russian. French summary) [MF 10182]

The author defines a natural measure in the space S of functions corresponding to states with a given energy value and shows that in the case of Maxwell-Boltzmann statistics physical quantities are very near their expected values except on a portion of S of small measure. There is no question of variation in time, so that the Schrödinger equation does not enter the discussion. Other statistics will be taken up in later papers.

J. L. Doob (Washington, D. C.)

Rosenfeld, L. Sur le comportement d'un ensemble canonique lors d'une transformation adiabatique. Nederl. Akad. Wetensch., Proc. 45, 970–972 (1942). [MF 10446]

When a dynamical system of statistical mechanics contains a parameter, and the parameter is varied adiabatically (that is, by small sudden steps so that no work is done on the system), it is conventionally assumed that, since $dQ=0$ (no heat imparted) so that $dQ/T=0$, there is no change of entropy. The author points out the deficiency of this reasoning. Inasmuch as the very concept of entropy in statistical mechanics is predicated on the assumption of a Gibbs-Boltzmann canonical distribution, it is necessary, before the proposition can have meaning, to show that the system, originally distributed canonically, remains so in the course of the adiabatic transformation. The proof that this is actually the case is the object of the paper. The background of the reasoning is the ergodic theorem. The proof is carried only to terms of the first order. *B. O. Koopman.*

Born, Max and Peng, H. W. Statistical mechanics of fields and the "apeiron." Nature 153, 164–165 (1944). [MF 10058]

The authors outline a new quantum theory of fields. The field in a finite volume Ω is regarded as a mechanical system described by its total energy and momentum. Each field component is considered as an operator (for the whole volume Ω) and is not regarded as a function of the coordinates at all. Certain commutation laws are given, from which the authors state that they have derived many of the classical results of quantum theory, but they indicate in outline that the new theory has fundamentally new attributes, both quantic and statistical. In the absence of proofs or adequately complete statements at the crucial points, it is not possible here to give either a physical or logical picture of the theory in any satisfactory detail. *B. O. Koopman.*

Onsager, Lars. Crystal statistics. I. A two-dimensional model with an order-disorder transition. Phys. Rev. (2) 65, 117–149 (1944). [MF 10089]

The partition function of a two-dimensioned "ferromagnetic" with scalar "spins" (Ising model) is computed rigorously for the case of a vanishing field. The spin of an atom entering the crystal lattice depends statistically on the spins of its neighbors. Operators which reverse the spins of one or more atoms form an algebra which is the direct product of n simple-quaternion algebras. These operators are studied in detail. The eigenwert problem involved in the computation of the partition function for a long strip crystal of finite width (n atoms), joined straight to itself around a cylinder, is thus solved by direct product decomposition. In the special case $n=\infty$, the sum is replaced by an integral, which is studied by formulas of hyperbolic trigonometry, and elliptic integrals. In particular, the "maximal" eigenvector and the maximal odd eigenvector are obtained. The choice of different interaction energies ($\pm J$, $\pm J'$) in the (0 1) and (1 0) directions does not complicate the problem. The two-way infinite crystal has an order-disorder transition at a temperature $T=T_c$ given by the condition

$$\sinh(2J/kT_c)\sinh(2J'/kT_c)=1.$$

The energy is a continuous function of T, but the specific heat becomes infinite as $-\log|T-T_c|$. For strips of finite width, the maximum of the specific heat increases linearly with $\log n$. The order-converting dual transformation invented by Kramers and Wannier effects a simple automorphism of the basis of the quaternion algebra which is natural to the problem in hand. In addition to the thermodynamic properties of the massive crystal, the free energy of a (0 1) boundary between areas of opposite order is computed; on this basis the mean ordered length of a strip crystal is

$$(\exp(2J/kT)\tanh(2J'/kT))^n.$$

J. S. Frame (East Lansing, Mich.).

Eisenschitz, R. Matrix theory of correlations in a lattice. I. Proc. Roy. Soc. London. Ser. A. 182, 244–259 (1944). [MF 10169]

Eisenschitz, R. Matrix theory of correlations in a lattice. II. Proc. Roy. Soc. London. Ser. A. 182, 260–269 (1944). [MF 10170]

The author summarizes his results as follows. Part I. "The statistical mechanics of some crystalline systems may be reduced to statistical correlations between objects which are the unit cells of a fictitious lattice. The correlations are deduced from postulates according to which some configurations of the cells are incompatible with some configurations of the neighbouring cells; if, on the other hand, configurations of neighbors are compatible with each other, their probabilities are to combine by multiplication. By these postulates matrices are implicitly defined such that the probability distribution for a chain of cells is found by forming the powers of a matrix. A similar approach to the statistics of a lattice involves infinite matrices. It does not seem practicable to give explicit expressions for these matrices. If appropriate conditions are complied with, the correlations in a chain are accounted for by adjusting the mean probability coefficients of the cells and for the rest regarding the cells as statistically independent. In this case the infinite matrices may be replaced by the outer power of finite matrices. As a result an equation is given by means of which the thermodynamical energy may be calculated as a function of temperature."

Part II. "The specific heat curve corresponding to order-disorder equilibrium is derived from the nearest neighbor model by means of the general theory of correlations in a lattice. For a lattice in two dimensions the resulting thermodynamic energy is an algebraic expression; for a lattice in three dimensions the thermodynamic energy is found by numerical methods. The c_v curve has a broad maximum and has no resemblance whatever to the experimental c_p curve. The reasons for this discrepancy are discussed."

B. O. Koopman (Washington, D. C.).

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AUTHOR INDEX

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